Counting the Number of *Unwanted* Eigenvalues

Intricate Challenges and the (Basically Simple) Symplectic Solution

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Microlocal and Global Analysis, Interactions with Geometry
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Dedicated to the memory of
ALBERTO CALDERÓN (14 Sep. 1920 – 16 Apr. 1998) and
ROBERT SEELEY (26 Feb. 1932 – 30 Nov. 2016)

Outline

Problem (Only too well-known engineering nuisance)

Data (Toy model)

A Linear self-adjoint and essentially negative operator in complex Hilbert space X with compact resolvent

$a_1 \leq \cdots \leq a_k$ Finite list of non-negative eigenvalues of A, $k \in \mathbb{N}$
Feedback, escalation risk, collapse

Theorem (Theoretical solution by two devices)

1. *Find* $\operatorname{spec}(A) \cap \mathbb{R}_+ =: \{a_1, \ldots, a_k\}$ *!*
2. $\forall_{1\leq j\leq k}$ *remove* a_j *by bounded perturbation* Π_j *!*

Comments. Ad (1): There doesn't exist a constructive method for PDEs, contrary to linear algebra and systems of linear ODEs.
Ad (2): Various elegant and practicable approaches in this conference, see also, e.g., the classic M. PEDERSEN, 1991, *Pseudodifferential perturbations and stabilization of distributed parameter systems: Dirichlet feedback control problems*, inspired by G. GRUBB.

Typical wave phenomena and *unwanted* eigenvalues I

Example (1. Fluid dynamics – Offshore floating wind technology with mooring lines)

Reducing risks from extreme waves and avoiding resonance

Task: control *coupling*

- wave propagation
- floater response

Validation

1. Detailed numerical flow simulation
2. Pool tests on small scale
3. Deep water tests on full scale

From P.D. TOMASELLI ET AL., Hybrid Modelling for Engineering Design of Floating Offshore Wind Turbine Foundations. Model Coupling and Validation, Trondheim 2019

Typical wave phenomena and *unwanted* eigenvalues II

Example (2. Electromagnetic waves – Stealth military technology)

F-117 Nighthawk of 1984, the first operational aircraft specifically designed around stealth technology

Stealth technologies: reducing reflection/emission of

- radar,
- infrared,
- visible light,
- radio frequency (RF) spectrum (and audio).

Promise of riskless ground attack and bombing.

Counting the *number* of unwanted eigenvalues

Theorem (Desuspension spectral flow formula)

$\mathrm{SF}\{A(s)_{\mathcal{D}(s)}\}_{0\leq s\leq 1} = -\mathrm{MAS}\{\gamma(\mathcal{D}(s)), \mathrm{CD}(A(s))\}_{0\leq s\leq 1}$, *admitting*

- *smooth variation of* $A(s)$ *and of Cauchy data space* $\mathrm{CD}(A(s))$,
- *continuous variation of Fredholm domain* $\mathcal{D}(s)$, *and*
- *demanding constant «ghosts» dimensions (or weak inner UCP).*
- *Range of validity to be explored further below.*

Corollary (Simple Morse index formula)

As above, A essentially negative with maximal domain $\mathcal{D}$ *and* $\mathbb{R}_+ \cap \mathrm{spec}(A) = \{a_1, \ldots, a_k\}$. *Set* $A(s) := (1-s)A - sI$ *for* $0 \leq s \leq 1$. *Then* $k = -\mathrm{SF}\{A(s)_{\mathcal{D}}\}_{0\leq s\leq 1} = \mathrm{MAS}\{\gamma(\mathcal{D}), \mathrm{CD}(A(s))\}_{0\leq s\leq 1}$.

Here $\gamma\colon \mathcal{D}_{\max} \longrightarrow \mathcal{D}_{\max}/\mathcal{D}_{\min}$ natural projection and

$\gamma(\mathcal{D}) := (\mathcal{D} \cap \mathcal{D}_{\max} + \mathcal{D}_{\min})/\mathcal{D}_{\min}$ for any linear subspace $\mathcal{D} \subset L^2(M; E)$.

NOTE: Long list of predecessors: Floer, Yoshida, Nicolaescu, Robbin & Salamon, Cappell & Lee & Miller, BBB & Furutani, Latushkin & collaborators, quite general in BBB, C. Zhu, *Memoirs Am. Math. Soc.* 2018, Thm. 4.5.4., p. 79.

Continuity of family of Calderón projections I

Data (Elliptic analysis, fix notation)

$A\colon C^\infty(M;E)\to C^\infty(M;F)$ linear elliptic differential op. of order d over smooth compact Riemannian manifold M of dimension n with boundary Σ, acting between sections of Hermitian vector bundles $E, F \to M$ with metric connections ∇^E, ∇^F, $E' := E|_\Sigma$, $F' := F|_\Sigma$.

$\gamma^j\colon C^\infty(M;E)\to C^\infty(\Sigma;E')$ jth jet trace operator with $\gamma^j u := (\nabla^E_\nu)^j u|_\Sigma$, ν inward unit normal field.

$\widetilde{\rho}_d := \Phi_d \circ \rho^d$ adjusted Cauchy trace operator with $\rho^d := (\gamma^0,\dots,\gamma^{d-1})$ and $\Phi_d := \mathrm{diag}(\Phi^{\frac{d-1}{2}},\dots)$ with $\Phi := (\Delta_\Sigma + 1)^{\frac{1}{2}}$

Lemma (Weak traces for elliptic operators)

$\widetilde{\rho}_d\colon H^s(M;E)\to H^{s-\frac{d}{2}}(\Sigma;E'^d)$ for $s > d-\frac{1}{2}$, and
$\widetilde{\rho}_d\colon \mathcal{D}(A_{\max})\to H^{-\frac{d}{2}}(\Sigma;E'^d)$ continuous.

Theorem (A. Calderón 1963, R.T. Seeley 1966,1969)

$\exists\, C(A)\colon C^\infty(\Sigma;E'^d)\to C^\infty(\Sigma;E'^d)$ pseudo-differential projection with $\mathrm{range}(C(A)) = \widetilde{\rho}_d(\ker A)$, $\quad \mathrm{range}(C_{-d/2}(A)) = \widetilde{\rho}_d(\ker A_{\max}) =: \mathrm{CD}(A)$, and $C_0(A)$ orthogonal.

Continuity of family of Calderón projections II

Main Theorem (BBB, J. DENG, Y. ZHOU, C. ZHU, arXiv:2012.03329)

• $(A_b)_{b\in B}$ family of elliptic diff. operators of order d, B top. space,
• $A_{b,s+\frac{d}{2}} : H^{s+\frac{d}{2}}(M;E) \to H^{s-\frac{d}{2}}(M;F)$ and
$A^t_{b,s+\frac{d}{2}} : H^{s+\frac{d}{2}}(M;F) \to H^{s-\frac{d}{2}}(M;E)$ make continuous families of bounded extensions in the respective operator norms for $s \geq \frac{d}{2}$,
• $J_{b,s}$ similar assumption for Green's forms,
• $C^{\mathrm{ort}}_s(A_b)$ L^2-orthogonalized Calderón projection, $s \in \mathbb{R}$,
• $\kappa_\pm(b) := \dim\ker A_{b,\min}$, resp., $\dim\ker A^t_{b,\min}$ do not depend on b,
where $A_{b,\min} : H^d_0(M;E) = \{h \in H^d(M;E) \mid \widetilde{\rho}^d(h) = 0\} \to L^2(M;F)$.
Then $\forall_{s\in\mathbb{R}}\, (C^{\mathrm{ort}}_s(A_b))_{b\in B}$ continuous family in operator norm of $H^s(\Sigma; E'^d)$ $\circlearrowleft$.

Note. BBB, G. CHEN, M. LESCH and C. ZHU, 2013, obtained the same result for $d = 1$, $s \geq -\frac{1}{2}$ and $\kappa_+(b) = \kappa_-(b) = 0$ for all $b \in B$. The proof was complicated.

Sketch of proof. *Let* $s \geq \frac{d}{2}$. Elementary functional analysis and use that the Cauchy trace map is bounded and surjective.

1. $\left(\ker A_{b,s+\frac{d}{2}}\right)_{b\in B}$ is continuous in $H^{s+\frac{d}{2}}(M;E)$.
2. $\left(\widetilde{\rho}^d(\ker A_{b,s+\frac{d}{2}})\right)_{b\in B}$ is continuous family of closed subsets in $H^s(\Sigma;E'^d)$ and so $\left(\operatorname{im} C_s^{\mathrm{ort}}(A_b)\right)_{b\in B}$.
3. $\left(C_s^{\mathrm{ort}}(A_b)\colon H^s(\Sigma;E'^d)\hookleftarrow\right)_{b\in B}$ continuous in the operator norm for all $s \geq \frac{d}{2}$, as claimed in Main Theorem!

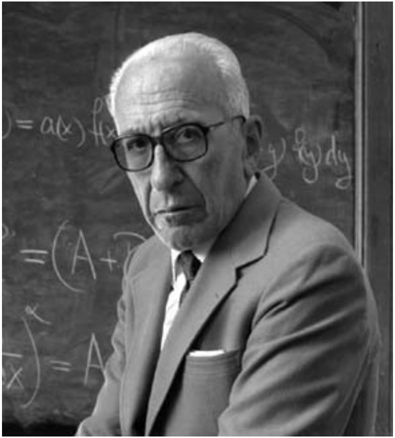
Let $s < \frac{d}{2}$. Extend solved case by duality and interpolation property of spaces and operators in Sobolev scales, yielding $\left(C_s^{\mathrm{ort}}(A_b)\colon H^s(\Sigma;E'^d)\hookleftarrow\right)_{b\in B}$ continuous family in the operator norm for all $s < \frac{d}{2}$.

UCP and Main Theorem *Sufficient condition*: Constant $\kappa_\pm(b)$ for all $b \in B$. *Necessary condition*: Homotopy invariance (HI) of the relative index $\operatorname{ind}(A_{b,C(b)}) = \operatorname{ind}(A_b, M\setminus\Sigma) := \kappa_+(b) - \kappa_-(b)$. So, our sufficient cond. may not be optimal.

Natural: $\left(B \text{ connected, HI, and } \exists_{b_0\in B} \text{ with } \kappa_+(b_0) = \kappa_-(b_0) = 0\right) \implies \left(\forall_{\mathbf{b}\in\mathbf{B}}\ (\kappa_+(\mathbf{b}) = \mathbf{0} \implies \kappa_-(\mathbf{b}) = \mathbf{0})\right)$, i.e., a **weak version of a Laurent Schwartz UCP Conjecture** of 1958. Warning: Circular argument

Legacy of A. CALDERÓN

- The theory of *singular* integral operators, developed jointly with A. ZYGMUND
- The Calderón projection
- The interpolation theory of pseudo-differential operators
- The Calderón Problem: Inverse (elliptic) problem for non-invasive imaging (e.g., Elect. Impedance Tomogr., concluding on medium from surface measurements)
- R. SEELEY, School of hard geometric analysis



Alberto Calderón, 1920-1998

Challenges (Wider perspective for future work)

1. Begin with engineering problem
2. Find energy minimizing integral
3. Derive Morse-Smale index form
4. Derive induced elliptic operator and domain
5. Apply multidimensional Morse Index Theorem à la SMALE 1965
6. Count the number of unwanted eigenvalues by applying the desuspension spectral flow formula

Inspiration S. SMALE, On the Morse index theorem, 1965
C. ZHU, A generalized Morse index theorem, 2006
F. DALBONO, A. PORTALURI, Morse-Smale index theorems for elliptic boundary deformation problems, 2012
G. COX, C. JONES, Y. LATUSHKIN, A. SUKHTAYEV, The Morse and Maslov indices for multidimensional Schrödinger operators..., 2016

Micro-local analysis in interaction with geometry, algebra and physics

Personal Experiences and General Impressions in Simplistic Terms

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Micro-local and Global Analysis, Interactions with Geometry
Potsdam 2D-Meeting, 21–25 February 2022

Informal 2D Discussion

Micro-local analysis (M-IA) in top-noch work

The seven demure, but mathematically lascivious sisters, Some of them *dispensable* (C.F. GAUSS) and *sportive exercises* (V. ARNOL'D)

1. 1746, Fundamental Theorem of Algebra; M-IA!
2. 1882, Transcendence of π: M-IA?
3. 1963, Continuum Hypothesis, independence of ZERMELO–FRAENKEL set theory: M-IA?
4. 1976, Four colours suffice: No M-IA
5. 1995, FERMAT's Last Theorem: A bit, but, sadly, mostly no M-IA
6. 2002, POINCARÉ Conjecture: M-IA!
7. ???, RIEMANN Hypothesis: If correct, for sure with M-IA

A student's paragon

- HIRZEBRUCH, RIEMANN-ROCH Theorem: Not much M-IA
- MILNOR, Exotic Spheres: Not much M-IA
- WALDHAUSEN, Knots, HEEGAARD Splitting: Not much M-IA

Aspects of M-IA interaction, in part rather simplistic

- In this panel, wording of mathematics: banned since GAUSS, anew exercised since HILBERT–COURANT–ATIYAH
- M-IA $\leftrightarrow$ Geometry: Geom. present in all mathematics
- M-IA $\leftrightarrow$ Algebra: Alg. present in all mathematics
- M-IA $\leftrightarrow$ Physics, and wider applications? MARC KAC:
  - *Pure math deals with deep questions in simple situations*. Pure math consists mainly of applications of appl. math; M-IA $\leftarrow$ Appl
  - *Appl. math deals with simple questions in extremely complicated models*. Recent advances in pure math contribute seldom to appl. math, no matter how *mathematically competent physics colleagues* are and how *persistent math colleagues* invoke physics terminology in grant applications; Appl $\not\leftarrow_{\text{i.g.}}$ M-IA
- Extending, explorative v. Consolidating, foundational, H. BOHR
- Architects v. Plumbers, I.M. SINGER V. R.T. SEELEY
- Constructivisme, G. VICO (*Verum Factum*) v. / & Deconstruction, J. DERRIDA (*Text is meaning*)

Table: Interactions between micro-local analysis (M-lA) with geometry, algebra, and/or physics: play a role in my teachers' and my own work Yes/No?

Terms/Topics	Classic math.	R-R	Exotic spheres	AS-I	UCP	Dirac, geom ops	SF	Morse	Free bprb exocytos.
Wording	No	No	No	Yes	No	No	No	Yes	Yes
Geometry	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Algebra	Gauss	Yes	Yes	Weak	No	On & off	Weak	Weak	No
M-lA $\leftarrow$ Appl	Yes	No	?	Shapira	Yes	Yes	Witten	Yes	Yes
Appl $\leftarrow$ M-lA	Seldom	No	No	Gauge ph.	No	Determnts	Materials	Not yet	Yes
Expanding, explorative	20th C.	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Consolidating, foundational	19th C.	Yes	Yes	No	No	Spec invar.	No	Yes	No
Architects	No	Grothendieck	No	Singer	Schwartz	Singer	Phillips	Arnold	Friedman
Plumbers	No	Hirzebruch	Milnor	Seeley	All	Grubb, Woj, Lesch	All	Furutani, Zhu et al.	D. Apush-kinskaya
Constructive	Yes	Yes	No	Yes	No	Yes	No	Yes	Yes
Deconstruct	No	No	Yes	No	Yes	Yes	Yes	No	No