Index of Callias-type operators and generalized spectral shift functions.

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Microlocal and Global Analysis, Interactions with Geometry,
Potsdam, February 21st, 2022

Introduction

One dimensional setup I

- Let $\mathcal{H}$ be a separable, complex Hilbert space.
- Let A_- be a self-adjoint operator in $\mathcal{H}$ with domain $\mathrm{Dom}\,(A_-)$.
- Let $A\,(x)$, $x \in \mathbb{R}$ be a family of self-adjoint operators in $\mathcal{H}$ with constant domain $\mathrm{Dom}\,(A\,(x)) = \mathrm{Dom}\,(A_-)$.
- The limiting operators $\lim_{x\to\pm\infty} A\,(x) = A_\pm$ exist in an appropriate sense and define a self-adjoint operator A_+ in $\mathcal{H}$ with domain $\mathrm{Dom}\,(A_+) = \mathrm{Dom}\,(A_-)$.

One dimensional setup II

- In $L^2(\mathbb{R}, \mathcal{H})$ define the derivative ∂

$$(\partial f)(x) := f'(x), \; f \in W^{1,2}(\mathbb{R}) \widehat{\otimes} \mathcal{H}.$$

- Define fibrewise multiplication $A(X)$, given by the family $A(x)$, $x \in \mathbb{R}$,

$$(A(X) f)(x) := A(x) f(x), \; f \in L^2(\mathbb{R}) \widehat{\otimes} \mathrm{Dom}(A_-).$$

- Together we obtain the operator $D = \partial + A(X)$ with domain

$$\mathrm{Dom}(D) = W^{1,2}(\mathbb{R}) \widehat{\otimes} \mathcal{H} \cap L^2(\mathbb{R}) \widehat{\otimes} \mathrm{Dom}(A_-).$$

Problems and known results I

- Under which assumptions on $A(\cdot)$ is D Fredholm and what is the Fredholm index of D, $\operatorname{ind}(D)$, expressed in terms of $A(\cdot)$?
- Firstly considered by M. Atiyah, V. Patodi and I. Singer in 1970s for $A(x)$ first order, elliptic, differential operators on compact, odd-dimensional manifolds with boundedly invertible asymptotic endpoints $A_{\pm}$.
- Further discussed by C. Callias (1978) in higher dimensions, but with $\mathcal{H}$ finite dimensional, stating a more general index formula than below.

Answer: The index=spectral flow-theorem, i.e. D is Fredholm, and

$$\operatorname{ind}(D) = \operatorname{sf}_{x\in\mathbb{R}}(A(x), 0).$$

Problems and known results II

- Observation: The spectra of $A(x)$ in $\mathcal{H}$ in the previous cases are necessarily all discrete!
- However: The spectral flow is not exclusively defined for families of discrete operators. For example J. Phillips (1996) showed that the spectral flow can be defined for families of Fredholm operators.
- Also, W. Müller (1988) showed that the spectral flow and the spectral shift function (SSF) coincide, if they both exist. This theme was further investigated by N. Azamov, A. Carey, P. Dodds, and F. Sukochev (2007)

A short introduction to spectral shift functions (SSF)

What is a spectral shift function?

- Introduced 1952 by J.Lifshitz while studying the S-matrix in quantum scattering problems.
- M.G. Krein gave a rigorous approach in 1953, stating his famous trace formula.
- Under certain trace class conditions on the perturbation pair B, A of self-adjoint operators in $\mathcal{H}$, we have for all $f \in C_c^\infty(\mathbb{R})$

$$\mathrm{tr}_{\mathcal{H}}\left(f\left(B\right) - f\left(A\right)\right) = \int_{\mathbb{R}} f'\left(\lambda\right)\xi\left(\lambda, B, A\right)\mathrm{d}\lambda,$$

  where $\xi\left(\cdot, B, A\right)$ is a fixed function, the SSF of B and A.
- Often, the SSF can be constructed using Stieltjes inversion on the logarithm of the perturbation determinant of B and A.

The Daletskii- S.Krein formula

- Consider the family of self-adjoint operators $A(t) = A + tT$, for T in $S^p(\mathcal{H})$ or in $B(\mathcal{H})$ and an "amenable" function $f : \mathbb{R} \to \mathbb{R}$, then

$$\frac{\mathrm{d}\,(f\,(A\,(t)))}{\mathrm{d}t} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f\,(\lambda) - f\,(\mu)}{\lambda - \mu} \mathrm{d}E_{A(t)}\,(\lambda)\, T\mathrm{d}E_{A(t)}\,(\mu)\,.$$

- The "operator differentiation" above makes sense in $S^2(\mathcal{H})$ for f Lipschitz. For other p and for $B(\mathcal{H})$, the measure on the right has to be amended and the store of amenable functions is smaller than Lipschitz, leading to the theory of double operator integrals (DOI). Its essential development is due to M. Birman, M. Solomyak and V. Peller.

The SSF appears again

- For $p = 1$ we may apply the trace to last formula. Integrated in t from 0 to 1 and using $T = B - A$, we obtain, commutating spectral measures in trace

$$\mathrm{tr}_{\mathcal{H}}\left(f\left(B\right) - f\left(A\right)\right) = \int_{\mathbb{R}} f'\left(\lambda\right) \mathrm{d}\Xi\left(\lambda\right),$$

$$\Xi\left(\sigma\right) = \int_0^1 \mathrm{tr}_{\mathcal{H}}\left(E_{A(t)}\left(\sigma\right) T\right) \mathrm{d}t.$$

- By M.Krein's trace formula, the finite measure Ξ has a density, namely the SSF $\xi\left(\cdot, B, A\right)$.

Problems and known results III

- A. Pushnitski (2008) considered families $A(x)$, which are absolutely continuous with derivative taking values in trace-class.
- F. Gesztesy, Y.Latushkin, K.Makarov, F.Sukochev and Y.Tomilov (GLMST) (2011) extended and exhausted Pushnitsky's approach for more general families $A(x)$.
- Fredholmness was given up in favor of considering families $A(x)$ of non-relative compact perturbations in thesis of O.F. (2021).

Problems and known results IV

Theorem[GLMST,2011]

Assume [GLMST] conditions (roughly: $A'(x)$ is integrable relative trace class with respect to A_-) then

- For $z \in \mathbb{C} \setminus [0, +\infty)$,

$$\operatorname{tr}_{L^2(\mathbb{R},\mathcal{H})}\left(\left(DD^* - z\right)^{-1} - \left(D^*D - z\right)^{-1}\right)$$
$$= \frac{1}{2z}\operatorname{tr}_{\mathcal{H}}\left(A_+\left(A_+^2 + z\right)^{-1/2} - A_-\left(A_-^2 + z\right)^{-1/2}\right).$$

- The SSF of (DD^*, D^*D) and (A_+, A_-) are related via Pushnitski's formula, for a.e. $\lambda > 0$:

$$\xi\left(\lambda, DD^*, D^*D\right) = \frac{1}{\pi}\int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} \frac{\xi\left(\mu, A_+, A_-\right)}{\sqrt{\lambda - \mu^2}} \mathrm{d}\mu.$$

Problems and known results V

Theorem[GLMST,2011]

If additionally $0 \in \rho(A_+) \cap \rho(A_-)$, D is Fredholm, and

$$\operatorname{ind}(D) = \lim_{\epsilon \searrow 0} \xi(\epsilon, DD^*, D^*D) = \xi(0, A_+, A_-)$$

$$= \pi^{-1} \lim_{\epsilon \searrow 0} \Im \left(\log \left(\det_{\mathcal{H}} \left((A_+ - i\epsilon)(A_- - i\epsilon)^{-1} \right) \right) \right),$$

with a branch of log, such that

$$\lim_{\Im(z) \to +\infty} \log \left(\det_{\mathcal{H}} \left((A_+ - z)(A_- - z)^{-1} \right) \right) = 0.$$

Problems and known results VI

Theorem[F,2021]

Under condition in [F] (roughly: $A'(x)$ is integrable relative trace class with respect to $A_-^{2-\epsilon}$) the following are true:

- For $t > 0$

$$\operatorname{tr}_{L^2(\mathbb{R},\mathcal{H})}\left(e^{-tDD^*} - e^{-tD^*D}\right)$$
$$= -\lim_{\epsilon \searrow 0} \operatorname{tr}_{\mathcal{H}}\left(e^{-\epsilon A_-^2}\left(\chi_{t,\epsilon}(A_+) - \chi_{t,\epsilon}(A_-)\right)e^{-\epsilon A_-^2}\right),$$

with $\chi_{a,b}(\lambda) := \frac{e^{b\lambda^2}}{2}\left(\operatorname{erf}\left((a+b)^{1/2}\lambda\right) - \operatorname{erf}\left(b^{1/2}\lambda\right)\right)$, and

$$\operatorname{tr}_{L^2(\mathbb{R},\mathcal{H})}\left(DD^*e^{-tDD^*} - D^*De^{-tD^*D}\right)$$
$$= (4\pi t)^{-1/2} \operatorname{tr}_{\mathcal{H}}\left(A_+e^{-tA_+^2} - A_-e^{-tA_-^2}\right).$$

Problems and known results VII

Theorem[F,2021]

Under the previous conditions the following are true:

- There is $\kappa \in \mathbb{R}$, such that for a.e. $\lambda > 0$

$$\xi\left(\lambda, DD^*, D^*D\right) = \kappa + \frac{1}{\pi}\int_{-\sqrt{\lambda}}^{+\sqrt{\lambda}} \frac{\xi\left(\mu, A_+, A_-\right)}{\sqrt{\lambda - \mu^2}} \mathrm{d}\mu.$$

- If $0 \in \rho\left(A_+\right) \cap \rho\left(A_-\right)$, then the Witten index of D exists, and

$$\mathrm{ind}_{W_s}\left(D\right) := \lim_{t \to +\infty} \mathrm{tr}_{L^2(\mathbb{R},\mathcal{H})}\left(e^{-tD^*D} - e^{-tDD^*}\right) = \kappa$$

$$= \lim_{t \to +\infty} \lim_{\epsilon \searrow 0} \mathrm{tr}_{\mathcal{H}}\left(e^{-\epsilon A_-^2}\left(\chi_{t,\epsilon}\left(A_+\right) - \chi_{t,\epsilon}\left(A_-\right)\right) e^{-\epsilon A_-^2}\right).$$

Setup in $\mathbb{R}^d$

General setup I

- Let $\mathcal{H}$ be a separable, complex Hilbert space.
- Let $r, d \in \mathbb{N}$ and consider ∇ on $\mathbb{R}^d$. Let $\left(c\left(\mathrm{d}x^j\right)\right)_{j=1}^{d} \in End\left(\mathbb{C}^r\right)$ be Clifford matrices. Define $\partial := ic\nabla$ on $W^{1,2}\left(\mathbb{R}^d, \mathbb{C}^r \otimes \mathcal{H}\right)$.
- Let A_0 be self-adjoint in $\mathbb{C}^r \otimes \mathcal{H}$, and define

$$\left(D_0 f\right)(x) := \left(\partial f\right)(x) + A_0 f(x),$$

for $f \in W^{1,2}\left(\mathbb{R}^d, \mathbb{C}^r \otimes \mathcal{H}\right) \cap L^2\left(\mathbb{R}^d, \mathrm{Dom}\left(A_0\right)\right)$.
- Let $A(x)$, $x \in \mathbb{R}^d$, be a family of self-adjoint operators with $\mathrm{Dom}\left(A(x)\right) = \mathrm{Dom}\left(A_0\right)$. Assume that there is a constant $c < \infty$, such that for all unit speed rays γ in $\mathbb{R}^d$,

$$\int_0^\infty \left\| \left(\nabla^{End(\mathbb{C}^r)}_{\dot{\gamma}(s)} A\right)(\gamma(s)) \right\|_{\mathrm{Dom}(A_0)^* \otimes \mathbb{C}^r \otimes \mathcal{H}} \mathrm{d}s \leq c.$$

General setup II

- Define

$$(Df)(x) := (\partial f)(x) + A(x) f(x),$$

for $f \in \mathrm{Dom}(D_0) \subset L^2(\mathbb{R}^d, \mathbb{C}^r \otimes \mathcal{H})$. We impose suitable conditions (regularity + uniform Kato Rellich bounds) on the family $A(\cdot)$, such that for a given $N \in \mathbb{N}$,

$$\mathrm{Dom}\left((DD^*)^N\right) = \mathrm{Dom}\left((D^*D)^N\right) = \mathrm{Dom}\left((D_0^*D_0)^N\right).$$

- Assume that for some $n \in \mathbb{N}$,

$$\sum_{j=1}^{d} \int_{\mathbb{R}^d} \left\| \nabla_{\partial_j}^{End(\mathbb{C}^r)} A(x) \left(1 + A_0^2\right)^{-\frac{n}{2}} \right\|_{S^1(\mathcal{H})} \mathrm{d}x < \infty,$$

$$\sum_{j=1}^{d} \int_{\mathbb{R}^d} \left\| \left[c\left(dx^j\right), A(x)\right] \left(1 + A_0^2\right)^{-\frac{n-1}{2}} \right\|_{S^1(\mathcal{H})} \mathrm{d}x < \infty.$$

Trace class properties

Theorem

Under the previous conditions

$$\left(1+(D^*D)^N\right)^{-1}-\left(1+(DD^*)^N\right)^{-1}\in S^1\left(L^2\left(\mathbb{R}^d,\mathbb{C}^r\otimes\mathcal{H}\right)\right),$$

for $N\geq n+d-1$. Also, for $t>0$, $k\in\mathbb{N}$,

$$(D^*D)^{kN}e^{-t(D^*D)^N}-(DD^*)^{kN}e^{-t(DD^*)^N}\in S^1\left(L^2\left(\mathbb{R}^d,\mathbb{C}^r\otimes\mathcal{H}\right)\right)$$

Idea of proof: Resolvent identities, then DOI theory applied to the function $w\mapsto\left(w^{-1}-1\right)^k e^{-t\left(w^{-1}-1\right)}\in C^\infty([0,1])$.

Trace formula

Conjecture (shown for $A(x)$ bounded)

Under the previous conditions

$$-\operatorname{tr}_{L^2(\mathbb{R}^d,\mathbb{C}^r\otimes\mathcal{H})}\left(\left(D^*D\right)^N e^{-t(D^*D)^N}-\left(DD^*\right)^N e^{-t(DD^*)^N}\right)$$

$$=\lim_{R\to\infty}\int_{S_R^{d-1}(0)}\operatorname{tr}_{\mathbb{C}^r\otimes\mathcal{H}}\left(ic\left(v^\flat\right)f_t^N\left(A\left(x\right)\right)\right)\mathrm{d}S\left(x\right)=\left(*\right),$$

where S is the induced measure on $S_R^{d-1}\subset\mathbb{R}^d$, v the outward pointing unit normal field, and

$$f_t^N\left(\lambda\right):=\left(2\pi\right)^{-d}\int_{\mathbb{R}^d}\lambda\left(\lambda^2+|\xi|^2\right)^{N-1}e^{-t\left(\lambda^2+|\xi|^2\right)^N}\mathrm{d}\xi.$$

Back to SSF and Daletskii-Krein

- The LHS can be reinterpreted as an integral similar to the Laplace transform of the SSF of the pair $\left((D^*D)^N, (DD^*)^N\right)$.
- The RHS can be rewritten as

$$(*) = \int_{\mathbb{R}^d} \mathrm{tr}_{\mathbb{C}^r \otimes \mathcal{H}} \left(ic\nabla^{End(\mathbb{C}^r)} \left(f_t^N (A) \right) (x) \right) \mathrm{d}x.$$

- DOI theory: $\nabla_v (f(A)) = \mathcal{J}_{\phi_f}^{E_A, E_A} (\nabla_v A)$, $\phi_f (x, y) = \frac{f(x)-f(y)}{x-y}$
- There exists a functional Ξ_A on Fourier transforms of finite measures, such that $(*) = \Xi_A \left[\left(f_t^N \right)' \right]$, given by

$$\Xi [\widehat{\mu}] = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_0^1 \mathrm{tr}_{\mathbb{C}^r \otimes \mathcal{H}} \left(ice^{-is\lambda A(x)} \nabla A(x) e^{-i(1-s)\lambda A(x)} \right) \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}\mu(\lambda).$$

Remarks and open questions

- Because $(*)$ is an integral over a divergence, we conclude that Ξ_A only depends on the values of $\lim_{|x|\to\infty} A(x)$.
- DOI theory implies that Ξ_A is local in the spectra of the involved operators, i.e. $\operatorname{supp} \Xi_A \subseteq \cup_{x\in\mathbb{R}^d}\sigma(A(x))$. Even more, if $\emptyset = B_\epsilon(\lambda) \cap \cup_{x\in\mathbb{R}^d}\sigma_{ess}(A(x))$, then Ξ_A is a constant in a neighbourhood of λ. Maybe one can restrict these conditions to $\lim_{|x|\to\infty} A(x)$ instead of all A?
- The conjectured trace formula leads to an Witten index formula like in [GLMST]. Under which additional assumptions is the index Fredholm?

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Sources II

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Thank you for your attention!