Heat kernel asymptotics for quaternionic contact manifolds

MICROLOCAL AND GLOBAL ANALYSIS, INTERACTIONS WITH GEOMETRY

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Outline

1. Definitions, examples and basic objects of subriemannian manifolds.
2. Known results.
3. Quaternionic contact manifolds.
4. The second heat invariant and scalar curvature.

Subriemannian manifolds

Definition

A subriemannian (shortly SR) manifold is a triple $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ where

(a) M is a connected orientable smooth manifold of dimension $n \geq 3$.

(b) $\mathcal{H}$ is a smooth distribution of constant rank $m < n$ which is bracket generating, i.e. if we set for $j \geq 1$

$$\mathcal{H}^1 := \mathcal{H} \text{ and } \mathcal{H}^{j+1} := \mathcal{H}^j + [\mathcal{H}, \mathcal{H}^j],$$

then for all $q \in M$ there is $s(q) \in \mathbb{N}$ such that $\mathcal{H}_q^{s(q)} = T_qM$.

(c) $\langle\cdot,\cdot\rangle$ is a Riemannian metric on $\mathcal{H}$.

$\mathcal{H}$ is called the horizontal distribution.
X is a horizontal vector field if X is tangential to $\mathcal{H}$.

Examples

(1) The (complex) Heisenberg group $\mathbb{H}_{2n+1}$:
As a manifold $\mathbb{H}_{2n+1} := \mathbb{C}^n \times \mathbb{R}$ and the group law is

$$(z,t) * (w,s) := (z + w, s + t + \frac{1}{2}\mathrm{Im}\langle z, w\rangle_{\mathbb{C}})$$

for $(z,t),(w,s) \in \mathbb{H}_{2n+1}$. The horizontal distribution $\mathcal{H}$ is generated by the left-invariant vector fields (in real coordinates x_j, y_j, t) :

$$X_j := \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t} \text{ and } Y_j := \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t} \text{ for } j = 1, \cdots, n$$

with non-trivial Lie brackets $[X_j, Y_j] = \frac{\partial}{\partial t}$.
Then the (rank $2n$) distribution $\mathcal{H}$ together with the Riemannian metric on $\mathcal{H}$ which makes $\{X_j, Y_j : j = 1, \cdots, n\}$ orthonormal, define a subriemannian structure on $\mathbb{H}_{2n+1}$.

Examples

(2) In the definition of the (complex) Heisenberg group, if we consider the quaternionic space $\mathbb{H}$ instead of $\mathbb{C}$, we obtain the quaternionic Heisenberg group $\mathbb{G}_{4n+3}$. Similarly, a subriemannian structure can be defined with a distribution of rank $4n$.

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(3) Other manifolds with natural subriemannian structures are: Carnot groups, contact manifolds, quaternionic contact manifolds, H-type foliations.

Basic objects in subriemannian geometry

- The (SR) distance of two points $d(x, y)$ in M is defined as the infimum of the lengths of absolutely continuous paths which are tangential to $\mathcal{H}$ and join x to y. By the *"Chow-Rashevskii"* theorem, this distance is always finite, i.e. we can always find a horizontal path joining two given points.

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- For each $q \in M$, there is a sequence of increasing subspaces:

$$\{0\} =: \mathcal{H}_q^0 \subset \mathcal{H}_q^1 \subset \cdots \subset \mathcal{H}_q^{r(q)-1} \subsetneq \mathcal{H}_q^{r(q)} = T_q M.$$

The integer $r(q)$ is called the step of the distribution at q.
A point $q \in M$ such that $(\dim(\mathcal{H}_p^j))_j$ is constant for p in a neighborhood of q is called regular. Otherwise, it is called singular.
$\Longrightarrow$ if all points are regular, then the SR manifold M is called equiregular.

Basic objects in subriemannian geometry

- At every regular point q, a nilpotent Lie algebra $\mathfrak{g}M(q)$ of step $r(q)$ can be defined:

$$\mathfrak{g}M(q) := \mathcal{H}_q^1 \oplus \mathcal{H}_q^2/\mathcal{H}_q^1 \oplus \cdots \mathcal{H}_q^{r(q)}/\mathcal{H}_q^{r(q)-1}.$$

  Note that this Lie algebra can also be endowed with an inner product in a canonical way. This algebra is called the tangential algebra at q.
- If M is equiregular, then there is a canonical smooth measure on M called *Popp measure*.

Basic objects in subriemannian geometry

- With respect to a smooth measure the sublaplacian is defined by the formula

$$\Delta^{\mu}_{sub} f := -\mathrm{div}_{\mu}(\mathrm{grad}_{\mathcal{H}} f) \text{ for } f \in C^{\infty}(M).$$

Here $\mathrm{grad}_{\mathcal{H}} f \in \mathcal{H}$ denotes the horizontal gradient defined by

$$X(f) = \langle \mathrm{grad}_{\mathcal{H}} f, X \rangle \text{ for } X \in \mathcal{H}.$$

If $\mu =$ the Popp measure, then the associated sublaplacian is called the intrinsic sublaplacian.
$\Longrightarrow$ The intrinsic sublaplacian is the most natural sublaplacian: it commutes with SR isometries!

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- if (M, d) is complete, we have a well-defined heat semi-group $(e^{-t\Delta^{\mu}_{sub}})_{t>0}$ on $L^2(M, \mu)$ (with smooth kernel K_t) induced by the self-adjoint extension of Δ^{μ}_{sub} on $L^2(M, \mu)$.

Classical problem in subriemannian geometry

Assume we have a SR manifold M together with a canonical connection ∇ which is compatible with the SR structure. Consider the small times asymptotics of the heat kernel $K_t(x, y)$ associated with the intrinsic sublaplacian Δ_{sub}:

$$K_t(x, x) = \frac{1}{t^{Q/2}}(c_0 + c_1 t + c_2 t^2 + \cdots).$$

Problem: What is the geometric interpretation of the heat invariants $c_0, c_1, \cdots$? Can we express these invariants with help of tensors induced by the given connection ∇ ?

Known results 1

(1) The Riemannian case: we have a canonical connection, the so-called *Levi-Civita connection*. The heat invariants can be expressed using components of the curvature tensor and their covariant derivatives. Furthermore, the second heat invariant c_1 equals the scalar curvature (up to a universal constant multiple).

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(2) R. Beals, P. Greiner & N. Stanton, 1984: If M is a CR manifold with certain symmetries, then the heat invariants can be expressed using components of the curvature and torsion tensors, and their covariant derivatives. Furthermore, the second heat invariant equals the scalar curvature of the Tanaka-Webster connection (up to a universal constant multiple).

Known results 2

(3) D. Barilari, 2011: If M is a 3D contact SR structure, then the second heat invariant equals the so-called *subriemannian curvature κ* (up to a universal constant multiple).
Geometric interpretation of the subriemannian curvature:

$$\frac{\mathrm{vol}(B(q,\epsilon))}{\mathrm{vol}(\widetilde{B}(0,\epsilon))} = 1 - c\kappa(q)\epsilon^2 + \text{correction term},$$

where c denotes a constant independent from M.

Quaternionic contact manifolds

We consider a class of SR manifolds where the SR structure is induced from a specific structure (O. Biquard, 2000):

Definition

A quaternionic contact manifold $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ is a $(4n+3)$-dimensional manifold M (with $n \geq 2$) together with a corank 3 distribution $\mathcal{H}$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$ such that

1. $\mathcal{H}$ is given locally as the kernel of an $\mathbb{R}^3$-valued 1-form $\eta = (\eta_1, \eta_2, \eta_3)$:
$$\mathcal{H} = \bigcap_{i=1}^{3} \mathrm{Ker}(\eta_i).$$
2. There are three almost complex structures I_1, I_2 and I_3 on $\mathcal{H}$ that satisfy the quaternionic commutation relations:
$$(I_i)^2 = I_1 I_2 I_3 = -\mathrm{Id} \text{ for } i = 1, 2, 3$$
with $2\langle I_i X, Y\rangle = d\eta_i(X, Y)$ for all horizontal vector fields $X, Y \in \mathcal{H}$ and $i = 1, 2, 3$.

Properties of qc manifolds

1. Qc manifolds are equiregular SR manifolds of step 2.
2. The tangential algebra at a given point is isomorphic to the quaternionic Heisenberg algebra.
3. For fixed $q \in M$, the horizontal space $\mathcal{H}_q$ is a left $\mathbb{H}$-module:

   $$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \cdot X := (a + bI_1 + cI_2 + dI_3)\, X$$

   for $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$ and $X \in \mathcal{H}_q$.
   $\Longrightarrow$ It is useful to work with an $\mathrm{Sp}(n)\mathrm{Sp}(1)$-frame.
4. The horizontal distribution $\mathcal{H}$ has a canonical complement and the metric $\langle \cdot, \cdot \rangle$ has a natural extension to TM.
5. The density of the Popp measure w.r.t. the Riemannian measure is constant.

The Biquard connection

Every qc manifold ($n > 1$) has a canonical connection ∇ called the Biquard connection.

Some properties of this connection are:

1. ∇ is a metric connection.
2. ∇ preserves the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$ and the 2-sphere bundle of almost complex structures $\mathbb{Q}$.
3. If $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ denotes the torsion tensor, then we have

$$T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V} \text{ and } T(\mathcal{V}, \mathcal{H}) \subset \mathcal{H}$$

On a qc manifold the scalar curvature κ is defined by considering only horizontal vector fields

$$\kappa := \sum_{\alpha,\beta} \langle R(X_\alpha, X_\beta)X_\beta, X_\alpha \rangle,$$

where $\{X_\alpha : \alpha = 1, \cdots, 4n\}$ denotes an orthonormal frame of $\mathcal{H}$ and

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

denotes the curvature tensor.

The main result

Theorem (A. Laaroussi)

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complete quaternionic contact manifold and let $q \in M$. Then, it holds:

$$K_t(q,q) = \frac{1}{t^{2n+3}} \left(c_0 + C_n \kappa(q) t + o(t) \right) \text{ as } t \to 0.$$

Here, c_0 is independent of the point q and C_n is a universal constant depending only on n and independent of the qc manifold M.

Application: If M is additionally compact, then the Popp volume $\mu(M)$ of M and $\int_M \kappa d\mu$ are spectral invariant.

Idea of the proof

In privileged coordinates $(x, z) \in \mathbb{R}^{4n+3}$ (centered at $q \in M$), the sublaplacian Δ_{sub} can be considered as a perturbation of the standard sublaplacian $\hat{\Delta}^q_{sub}$ on the quaternionic Heisenberg group $\mathbb{G}(q) \simeq \mathbb{G}_{4n+3}$:

$$\Delta_{sub} = \hat{\Delta}^q_{sub} + \mathcal{A}_{-1}(q) + \mathcal{A}_0(q) + \cdots$$

where $\mathcal{A}_j(q)$ is a homogeneous differential operator, of degree j w.r.t. the parabolic dilations δ_λ defined in privileged coordinates (x, z) by

$$\delta_\lambda(x, z) := (\lambda x, \lambda^2 z).$$

Due to a result of Y. C. de Verdière, L. Hillairet & E. Trélat (2020), the second heat invariant can be expressed in the form

$$c_1(q) = \int_0^1 \int_{\mathbb{G}(q)} \hat{K}_{1-s}(0, \xi) \mathcal{A}_0 \left(\hat{K}_s(\xi, 0) \right) d\xi ds$$

where $\hat{K}$ denotes the heat kernel on the quaternionic Heisenberg group $\mathbb{G}_{4n+3}$.

Idea of the proof

Using a local orthonormal frame $X_1, \cdots, X_{4n}$ of the horizontal distribution $\mathcal{H}$ the intrinsic sublaplacian Δ_{sub} can be expressed locally

$$\Delta_{sub} = -\left(\sum_{\alpha=1}^{4n} X_\alpha^2 + \mathrm{div}_\mu(X_\alpha) X_\alpha \right).$$

Hence, to compute the parabolic expansion of the sublaplacian it suffices to compute the parabolic Taylor expansion of the horizontal frame (in privileged coordinates (x, z))

$$X_\alpha = X_\alpha^{(-1)} + X_\alpha^{(0)} + X_\alpha^{(1)} + \cdots$$

The homogeneous differential operator $\mathcal{A}_0$ has the expression

$$\mathcal{A}_0 = \sum_\alpha X_\alpha^{(-1)} X_\alpha^{(1)} + X_\alpha^{(1)} X_\alpha^{(-1)} + \left(X_\alpha^{(0)} \right)^2.$$

Idea of the proof

The next step is to construct privileged coordinates on which the homogeneous term $\mathcal{A}_0(q)$ is given by components of the curvature and torsion tensors w.r.t. to a special frame. These coordinates are called parabolic normal coordinates (C. Kunkel, 2008).

$\Longrightarrow$ the second heat invariant $c_1(q)$ can be expressed with help of components of the curvature and torsion tensors of the Biquard connection w.r.t. an adapted frame at q.

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Problem: How to simplify the expression of $c_1(q)$?
$\Longrightarrow$ Instead of using invariant theory of $\mathrm{Sp}(n)\mathrm{Sp}(1) \times O(3)$, we use symmetries and exponential decay of the heat kernel $\hat{K}$ on the quaternionic Heisenberg group. It turns out that $c_1(q)$ is a linear combination of traces of some Ricci-type and torsion tensors.

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$\Longrightarrow$ Finally, we use standard identities involving the curvature and torsion tensors to obtain our result.

Thank you for your attention!

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