# Elliptic Sequences

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## Outline.

I'll discuss some aspects of sequences of first order differential operators

$$\cdots \to C^\infty(M;E^{q-1}) \xrightarrow{\mathbb{P}_{q-1}} C^\infty(M;E^q) \xrightarrow{\mathbb{P}_q} C^\infty(M;E^{q+1}) \to \cdots,$$

elliptic in the sense that the symbol sequence is exact at any nonzero covector $\boldsymbol{\xi} \in T^*M$,

$$\cdots \to E^{q-1}_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_{q-1})(\boldsymbol{\xi})} E^q_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_q)(\boldsymbol{\xi})} E^{q+1}_{\pi(\boldsymbol{\xi})} \to \cdots$$

$M$ is a manifold and the $E^q$ are complex vector bundles.

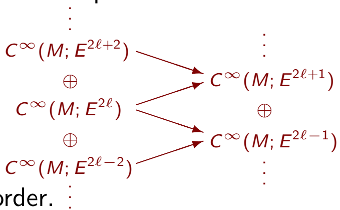
but the the composition $\mathbb{P}_{q+1} \circ \mathbb{P}_q$ is not necessarily zero, so not necessarily a complex.

Some features are similar to those of elliptic complexes, but some are not. For instance, if $M$ is compact, the vector bundles $E^q$ are Hermitian, $\mathfrak{m}$ is a smooth measure on $M$, then the index of the rolled up sequence is equal to

$\sum_q (-1)^q \dim\ker \square_q$,

where $\square_q = \mathbb{P}_q^\star \mathbb{P}_q + \mathbb{P}_{q-1}\mathbb{P}_{q-1}^\star$,

(This in itself is not deep.)




Comment: Because of the exactness of the symbol sequence, the operators $\mathbb{P}_{q+1} \circ \mathbb{P}_q$ are at most first order.

## By way of motivation...

One of the general problem one would like to figure out is the validity of Poincaré lemma for first order elliptic differential complexes. This is the following problem:

Suppose you have a complex

$\mathbb{P}_{q+1} \circ \mathbb{P}_q = 0$ for all $q$

$$\cdots \to C^\infty(M; E^{q-1}) \xrightarrow{\mathbb{P}_{q-1}} C^\infty(M; E^q) \xrightarrow{\mathbb{P}_q} C^\infty(M; E^{q+1}) \to \cdots$$

which is elliptic: the symbol sequence

$$\cdots \to E^{q-1}_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_{q-1})(\boldsymbol{\xi})} E^q_{\pi(\boldsymbol{\xi})} \xrightarrow{\boldsymbol{\sigma}(\mathbb{P}_q)(\boldsymbol{\xi})} E^{q+1}_{\pi(\boldsymbol{\xi})} \to \cdots$$

is exact at any $\boldsymbol{\xi} \in T^*M \backslash 0$.

The problem is, given $x_0 \in M$ and some $q_0$, determine whether the problem

$$\mathbb{P}_{q_0-1} u = f,\ f \text{ a section of } E_{q_0} \text{ defined near } x_0 \text{ with } \mathbb{P}_{q_0} f = 0$$

always has a solution $u$, a section of $E_{q_0-1}$, defined near $x_0$.

This holds for any $x_0$ and any $q_0$ when the complex is defined by an elliptic structure (this is a consequence of the Dolbeault lemma and the Poincaré lemma for the de Rham complex). Beyond this, a complete answer is known for complexes defined by an involutive structure of rank 1 (Cordaro-Hounie, Treves).

Tarkhanov has a book on differential complexes where he states a theorem giving Poincaré lemma as above assuming an ad-hoc pseudoconvexity condition to establish an estimate à la Kohn. Note that the work of Spencer and others strives for equivalent formulations of the problem, not to a solution. What follows is part of work in progress.

**Example.** With $M$ a neighborhood of 0 in $\mathbb{C}$, let

$$0 \to C^\infty(M;\mathbb{C}) \xrightarrow{\mathbb{P}_0} C^\infty(M;\mathbb{C}\oplus\mathbb{C}) \xrightarrow{\mathbb{P}_1} C^\infty(M;\mathbb{C}) \to 0$$

with operators

$$\mathbb{P}_0(f) = (z^k\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \overline{z}}), \quad \mathbb{P}_1(\phi_1,\phi_2) = \frac{\partial \phi_1}{\partial \overline{z}} - z^k\frac{\partial \phi_2}{\partial z}$$

$k > 0$ integer

$$\mathbb{P}_1\mathbb{P}_0 f = (z^k f_z)_{\overline{z}} - z^k (f_{\overline{z}})_z = 0$$

Ellipticity: With $\nu = \xi dx + \eta dy \neq 0$ and $\zeta = \xi + i\eta$,

$\mathbb{C} \ni f \mapsto \boldsymbol{\sigma}(\mathbb{P}_0)(\nu)(f) = (i\zeta z^k f, i\overline{\zeta} f)$ is injective

$\mathbb{C}\oplus\mathbb{C} \ni (\phi_1,\phi_2) \mapsto \boldsymbol{\sigma}(\mathbb{P}_1)(\zeta)(\phi_1,\phi_2) = i\overline{\zeta}\phi_1 - iz^k\zeta\phi_2$ is surjective

If $i\overline{\zeta}\phi_1 - iz^k\zeta\phi_2 = 0$, then $\phi_1 = z^k\zeta\phi_2/\overline{\zeta}$. With $f = \phi_2/(i\overline{\zeta})$ get $\boldsymbol{\sigma}(\mathbb{P}_0)(\zeta)(f) = (\phi_1,\phi_2)$

But: $\mathbb{P}_1(\phi_1,\phi_2) = 0$ is not sufficient for solvability:
It also requires $\phi_1 = z^k\psi$ for some $\psi$.

$\frac{\partial \phi_1}{\partial \overline{z}} = z^k \frac{\partial \phi_2}{\partial z}$ implies

$$\phi_1 = z^k\psi + \sum_{\ell=0}^{k-1} a_\ell z^\ell$$

with $\frac{\partial \psi}{\partial \overline{z}} = \frac{\partial \phi_2}{\partial z}$

The correct statement of the problem is:

For any $f$ near 0 such that $\mathbb{P}_q f = 0$, find $u$ near $x_0$ such that $\mathbb{P}_{q-1} u = f$, provided the equation is formally solvable at $x_0$ ...as is well known...

$$z^k\frac{\partial f}{\partial z} = z^k\psi + \sum_{\ell=0}^{k-1} a_\ell z^\ell ?$$

Idealy: try to get solutions via analysis...

By the way, an elliptic
$0 \to C^\infty(M,E) \xrightarrow{\mathbb{P}_0} C^\infty(M,F) \to 0$
is always locally solvable

**Example.** An elliptic structure on $M$ is an involutive subbundle $\mathcal{V} \hookrightarrow \mathbb{C}TM$ such that

Involutive: $X, Y \in C^\infty(M; \mathcal{V}) \implies [X, Y] \in C^\infty(M; \mathcal{V})$

(*) $$\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}TM.$$

Associated to $\mathcal{V}$ there is an elliptic **complex** with operators

(**) $$\cdots \to C^\infty(M; \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_q} (M; \textstyle\bigwedge^q \overline{\mathcal{V}}^*) \to \dots$$

and principal symbol

$$\boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})(\phi) = \sqrt{-1}\, \iota^*(\boldsymbol{\xi}) \wedge \phi$$

$\iota^* : \mathbb{C}T^*M \to \overline{\mathcal{V}}^*$ is the dual of the inclusion $\iota : \overline{\mathcal{V}} \hookrightarrow \mathbb{C}TM$

Using the Newlander-Nirenberg theorem one can find local coordinates

$$t_1, \dots, t_\kappa,\ x_1, y_1, \dots, x_m, y_m$$

$z_j = x_j + i y_j$

near any point such that $\overline{\mathcal{V}}$ is spanned by $\partial_{t_1}, \dots, \partial_{t_\kappa}, \partial_{\overline{z}_1}, \dots \partial_{\overline{z}_m}$.

This complex is an amalgam of the de Rham and Dolbeault complexes

Say that $\mathcal{V} \subset \mathbb{C}TM$ is almost elliptic if (*) holds with $\mathcal{V}$ not necessarily involutive. I'll give examples of elliptic sequences similar to (**) later. What one would like to have is some way of detecting that an elliptic complex is a combination of simpler complexes. I'll give a theorem in in this direction. First, some more examples

**Example.** Suppose $\mathcal{V}_1$ and $\mathcal{V}_2$ are two possibly unrelated elliptic structures on $M$. Each defines a complex

$$\cdots \to C^\infty(M;\textstyle\bigwedge^{q-1}\overline{\mathcal{V}}_i^*) \xrightarrow{\mathbb{D}_{q-1}} C^\infty(M;\textstyle\bigwedge^{q}\overline{\mathcal{V}}_i^*) \xrightarrow{\mathbb{D}_{q}} C^\infty(M;\textstyle\bigwedge^{q+1}\overline{\mathcal{V}}_i^*) \to \cdots$$

One can the put these together as a shifted direct sum with

$$\begin{matrix}\mathbb{D}_{q-1} \\ \oplus \\ \mathbb{D}_{q-1+\ell}\end{matrix} : \begin{matrix} C^\infty(M;\bigwedge^{q-1}\overline{\mathcal{V}}_1^*) \\ \oplus \\ C^\infty(M;\bigwedge^{q+\ell-1}\overline{\mathcal{V}}_2^*)\end{matrix} \to \begin{matrix} C^\infty(M;\bigwedge^{q}\overline{\mathcal{V}}_1^*) \\ \oplus \\ C^\infty(M;\bigwedge^{q+\ell}\overline{\mathcal{V}}_2^*)\end{matrix}$$

with some fixed integer $\ell$. The Poincaré lemma holds in any degree via estimates, but if you don't know the nature of the complex this is very hard to determine if you don't already know the structure of the complex.

So try to get a sense of what elliptic complexes can be, to learn how to extract pseudoconvexity condition to get solvability via estimates

The condition of being a complex, $\mathbb{P}^2 = 0$, is an analytic condition. Try first dropping it, keeping only exactness of the symbol sequence (which is ellipticity)

**Comment.** Suppose $\mathcal{V} \subset \mathbb{C}TM$ is a subbundle with corank 1, $\mathbb{C}TM/\mathcal{V}$ has rank 1. Suppose $\mathcal{V}^{\perp}$ is locally spanned by $dZ$ with some smooth $Z$. For the complex associated to $\mathcal{V}$,

"$\mathcal{V}$ is locally integrable"

$$\cdots \to C^{\infty}(M; \bigwedge^{q-1}\overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_{q-1}} C^{\infty}(M; \bigwedge^{q}\overline{\mathcal{V}}^*) \xrightarrow{\mathbb{D}_{q}} C^{\infty}(M; \bigwedge^{q+1}\overline{\mathcal{V}}^*) \to \cdots$$

which is not elliptic, there are known necessary and sufficient conditions for the validity of the Poincaré lemma. They depend on the degree $q$, and are a generalization of condition $P$ of Nirenberg-Treves.

(Treves, Cordaro & Hounie)

No general conditions are known for higher corank.

Cordaro & Hounie:
Suppose that $Z \in C^{\infty}$ and $1 \leq q \leq n$.
Then condition $(\mathcal{P})_{q-1}$ implies local solvability at $x_0$ in degree q.

$(\mathcal{P})_q$: Given any neighborhood $U$ of $x_0$ in $M$ there is another neighborhood $V \subset U$ of $x_0$ such that, for every regular value $z_0 \in \mathbb{C}$ of $Z : U \to \mathbb{C}$, either

$$\mathcal{F}(z, V) = \emptyset$$

$\mathcal{F}(V, z) = \{x \in V : Z(x) = z\}$

or else the homomorphism

$$\tilde{H}_q(\mathcal{F}(V, z)) \to \tilde{H}_q(\mathcal{F}(U, z))$$

(augmented homology with coefficients in $\mathbb{C}$)

induced by the inclusion map $\mathcal{F}(V, z) \to \mathcal{F}(V, z)$ vanishes identically.

From the paper by Cordaro & Hounie

**Example.** Familiar construction using connections. Suppose $E$ is a vector bundle, with connection $\nabla_1 : C^\infty(M; E) \to C^\infty(M; T^*M \otimes E)$. Define

Back to elliptic sequences

$$\cdots \to C^\infty(M;{\textstyle\bigwedge}^{q-1}M\otimes E) \xrightarrow{\nabla_{q-1}} C^\infty(M;{\textstyle\bigwedge}^{q}M\otimes E) \xrightarrow{\nabla_{q}} C^\infty(M;{\textstyle\bigwedge}^{q+1}M\otimes E) \to \cdots$$

by

$$\nabla_q(\eta \otimes \phi) = (-1)^q \phi \wedge \nabla\eta + (d\phi)\otimes\eta.$$

Then $\nabla_{q+1} \circ \nabla_q = \Omega_q$ is an operator of order 0, the curvature of $\nabla$. Since

$$\boldsymbol{\sigma}(\nabla_q)(\boldsymbol{\xi})(\eta \otimes \alpha) = \sqrt{-1}\,\boldsymbol{\xi} \wedge \eta \otimes \alpha$$

this is an elliptic sequence.

This motivates more examples.

In the example above, $\mathcal{V} = \mathbb{C}TM$. Replace $\mathbb{C}TM$ by an almost elliptic subbundle $\mathcal{V}$

**Example.** Suppose $\mathcal{V} \hookrightarrow \mathbb{C}TM$ is an almost elliptic structure on $M$. Pick a connection $\nabla$ on $\overline{\mathcal{V}}^*$. For sections $\theta_1, \dots, \theta_q$ of $\overline{\mathcal{V}}^*$ let

$\mathcal{V} + \overline{\mathcal{V}} = \mathbb{C}TM$

$$\mathbb{D}_q(\theta_1 \wedge \cdots \wedge \theta_q) = \mathrm{Alt} \circ (\iota^* \otimes \mathrm{I}) \sum_{j=1}^{q} (-1)^{j+1} (\nabla \theta^j) \wedge \theta^1 \wedge \cdots \wedge \hat{\theta}^j \wedge \cdots \wedge \theta^q.$$

This defines the operators $\mathbb{D}_q : C^\infty(M; \bigwedge^q \overline{\mathcal{V}}^*) \to C^\infty(M; \bigwedge^{q+1} \overline{\mathcal{V}}^*)$ of an elliptic sequence, with principal symbols

$\mathbb{D}_0 f = \iota^* df.$

$$\boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})(\eta) = \sqrt{-1}\iota^*(\boldsymbol{\xi}) \wedge \eta, \quad \boldsymbol{\xi} \in T^*M,\ \eta \in \bigwedge^q \overline{\mathcal{V}}^*_{\pi(\boldsymbol{\xi})}$$

Similarly one can define

$$\cdots \to C^\infty(M; \bigwedge^{q-1} \overline{\mathcal{V}}^* \otimes E) \xrightarrow{\mathbb{D}_{q-1}} C^\infty(M; \bigwedge^{q} \overline{\mathcal{V}}^* \otimes E) \xrightarrow{\mathbb{D}_{q}} C^\infty(M; \bigwedge^{q+1} \overline{\mathcal{V}}^* \otimes E) \to \cdots$$

with

$$\boldsymbol{\sigma}(\mathbb{D}_q)(\boldsymbol{\xi})(\eta \otimes \alpha) = \sqrt{-1}\, \iota^*(\boldsymbol{\xi}) \wedge \eta \otimes \alpha, \quad \boldsymbol{\xi} \in T^*M,\ \eta \in \bigwedge^q \overline{\mathcal{V}}^*_{\pi(\boldsymbol{\xi})},\ \alpha \in E_{\pi(\boldsymbol{\xi})}.$$

Let

$$d_q(\zeta)(\eta \otimes \alpha) = \sqrt{-1}\iota^* \zeta \wedge \eta \otimes \alpha.$$

"Standard principal symbol."

$\mathbb{P}_q$ is a $1^{\text{st}}$ order differential operator, so $\boldsymbol{\xi} \to \boldsymbol{\sigma}(\mathbb{P}_q)(\boldsymbol{\xi}) \in \mathrm{Hom}(E^q, E^{q+1})$ is linear in $\boldsymbol{\xi}$. Extend to $\mathbb{C}T^*M \to \mathrm{Hom}(E^q, E^{q+1})$.

Note 1: $\overline{\mathcal{V}}$ is the annihilator of the kernel of $\iota^* : \mathbb{C}T^*M \to \overline{\mathcal{V}}^*$

Note 2: $\mathbb{D}^2$ may not be zeroth order, but just first order.

**Proposition.** Let $\mathcal{V} \subset \mathbb{C}TM$ be a subbundle of rank $m$, let $E^0, \dots, E^m$ be complex vector bundles over $M$, $\operatorname{rk} E^0 = \operatorname{rk} E^m$. Suppose the homomorphisms

$$p_q : \mathbb{C}T^*M \to \operatorname{Hom}(E^q, E^{q+1}), \quad q = 0, 1, \dots, m-1$$

satisfy:

(a) for each $\zeta \in \mathbb{C}T^*M$, $p_{q+1}(\zeta) \circ p_q(\zeta) = 0$

(b) $\ker p_q = \overline{\mathcal{V}}^\perp$.

(c) For $q \geq 0$, the map $E^q \otimes \mathbb{C}T^*M \to E^{q+1}$ induced by the bilinear map $(\zeta, \alpha) \mapsto p_q(\zeta)(\alpha)$ is surjective.

Then there are unique isomorphisms $\Phi_q : E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^q$, $\Phi_0 = \mathrm{I}$, such that

$$\Phi_{q+1}^{-1} \circ p_q(\zeta) \circ \Phi_q = d_q(\zeta) \quad \zeta \in \mathbb{C}T^*M$$

with $d_q(\zeta)(\eta \otimes \alpha) = \sqrt{-1}\iota^*\zeta \wedge \eta \otimes \alpha$.

Use $\overline{\mathcal{V}}^* \approx \mathbb{C}TM/\overline{\mathcal{V}}^\perp$, so view the $p_q$ as $p_q : \overline{\mathcal{V}}^* \to \operatorname{Hom}(E_q, E_{q+1})$. Define

$$\hat{\Phi}_{q,\ell} : \bigoplus_{i=1}^{\ell} \overline{\mathcal{V}} \otimes E^q \to E^{q+\ell}$$

by

$$\hat{\Phi}_{q,\ell}(\zeta_\ell \otimes \cdots \otimes \zeta_1 \otimes \alpha) = p_{q+\ell-1}(\zeta_\ell) \circ p_{q+\ell-2}(\zeta_{\ell-1}) \circ \cdots \circ p_q(\zeta_1)(\alpha)$$

This vanishes if two successive $\zeta$'s are equal. It gives a map

$$\Phi_{q,\ell} : \bigwedge^\ell \overline{\mathcal{V}}^* \otimes E^q \to E^{q+\ell}$$

such that

$$\Phi_{q,\ell+m}(\zeta \wedge \eta \otimes \alpha) = -\Phi_{q+m,\ell}(\zeta) \circ \Phi_{q,m}(\eta \otimes \alpha).$$

$\eta = \eta_m \wedge \cdots \wedge \eta_1$

$\zeta = \zeta_\ell \wedge \cdots \wedge \zeta_1$

In particular $\Phi_{0,q+1}(\zeta_1 \wedge \eta \otimes \alpha) = \Phi_{q,1}(\zeta_1) \circ \Phi_{0,q}(\eta \otimes \alpha)$.

This is $\Phi_{0,q+1} \circ d_q(\zeta_1) = p_q(\zeta_1) \circ \Phi_{0,q}$.

Each $\Phi_{0,q}$ is surjective, so

$$\Phi_{0,m} : \bigwedge^m \overline{\mathcal{V}}^* \otimes E^0 \to E^m \text{ is surjective.}$$

Since

$$\operatorname{rk}(\bigwedge^m \overline{\mathcal{V}}^* \otimes E^0) = \operatorname{rk} E^m,$$

$\Phi_{0,m}$ is an isomorphism.

Then $\Phi_{0,q}$ is an isomorphism for each $q$.

$\mathcal{V} \subset \mathbb{C}TM$ of rank $m$, $E^0, \dots, E^m \to M$ complex vector bundles, $\operatorname{rk} E^0 = \operatorname{rk} E^m$.

$p_q : \mathbb{C}T^*M \to \operatorname{Hom}(E^q, E^{q+1})$, $q = 0, \dots, m-1$

satisfy:

(a) $\forall \zeta \in \mathbb{C}T^*M$, $p_{q+1}(\zeta) \circ p_q(\zeta) = 0$

(b) $\ker p_q = \overline{\mathcal{V}}^\perp$.

(c) the maps $E^q \otimes \mathbb{C}T^*M \to E^{q+1}$, $(\zeta \otimes \alpha) \mapsto p_q(\zeta)(\alpha)$ are surjective.

$\exists!$ isomorphisms $\Phi_q : E^0 \otimes \bigwedge^q \overline{\mathcal{V}}^* \to E^q$ such that

$$\Phi_{q+1}^{-1} \circ p_q(\zeta) \circ \Phi_q = d_q(\zeta) \quad \zeta \in \mathbb{C}T^*M$$

with $d_q(\zeta)(\eta \otimes \alpha) = \sqrt{-1}\, \iota^*\zeta \wedge \eta \otimes \alpha$. ($\Phi_0 = \mathrm{I}$).

**Proposition.** Let $E^0, \dots, E^m$ be complex vector bundles over $M$ with $\operatorname{rk} E^0 = \operatorname{rk} E^m$. Suppose

$$\cdots \xrightarrow{\mathbb{P}_{q-1}} C^\infty(M; E^q) \xrightarrow{\mathbb{P}_q} C^\infty(M; E^{q+1}) \xrightarrow{\mathbb{P}_{q+1}} \cdots$$

is an elliptic sequence of first order differential operators. Let $p_q = \boldsymbol{\sigma}(\mathbb{P}_q)$, a homomorphism $\mathbb{C}T^*M \to \operatorname{Hom}(E^q, E^{q+1})$

$\boldsymbol{\sigma}(\mathbb{P}_{q+1})(\zeta) \circ \boldsymbol{\sigma}(\mathbb{P}_q)(\zeta) = 0$
for each $\zeta \in \mathbb{C}T^*M$

Suppose

(a) There is a subbundle $\mathcal{V} \subset \mathbb{C}TM$ of rank $m$ such that $\overline{\mathcal{V}}^\perp = \ker \boldsymbol{\sigma}(\mathbb{P}_q)$ for all $q$.

(b) The maps $\mathbb{C}T^*M \otimes E^q \to E^{q+1}$ induced by $(\zeta, \alpha) \mapsto p_q(\zeta)(\alpha)$ are surjective.

Then $\mathcal{V}$ is an almost elliptic structure.

Furthermore, if $\mathbb{P}_{q+1} \circ \mathbb{P}_q$ has of order 0 for some $q < m$, then $\mathcal{V}$ is involutive.

## Some analysis: Domains, cohomology.

We now assume $M$ compact, the vector bundles $E^q \to M$ are Hermitian and $\mathfrak{m}$ is a smooth measure. The sequence

$$\cdots \to C^\infty(M; E^{q-1}) \xrightarrow{\mathbb{P}_{q-1}} C^\infty(M; E^q) \xrightarrow{\mathbb{P}_q} C^\infty(M; E^{q+1}) \to \cdots$$

is elliptic and $\mathbb{P}_{q+1} \circ \mathbb{P}_q = a_q$ is of order 0, $a_q : E^q \to E^{q+2}$ is a vector bundle homomorphism. Define

$$\mathcal{D}^q = \{u \in L^2(M; E^q) : \mathbb{P}_q u \in L^2(M; E^{q+1})\}$$
$$\mathcal{D}^{q,\star} = \{u \in L^2(M; E^{q+1}) : \mathbb{P}_q^\star u \in L^2(M; E^q)\}$$

"The maximal domain is the closure of $C^\infty(M; E^q)$"

Then

$$u \in \mathcal{D}^q \implies \mathbb{P}_q u \in L^2(M; E^{q+1}) \text{ and } a_q u \in L^2(M; E^{q+2}).$$

So $\mathbb{P}_{q+1} u \in \mathcal{D}^{q+1}$. Get a sequence

$$\cdots \to \mathcal{D}^{q-1} \xrightarrow{\mathbb{P}_{q-1}} \mathcal{D}^q \xrightarrow{\mathbb{P}_q} \mathcal{D}^{q+1} \to \cdots$$

(which may not be a complex).

Ask:

Given $f$ such that $\mathbb{P}_q f \in \operatorname{rg} a_{q-1}$, find $v$ such that $\mathbb{P}_{q-1} v = f$.

**Proposition**. For the sequence

$$(*) \qquad \cdots \to \mathcal{D}^{q-1} \xrightarrow{\mathbb{P}_{q-1}} \mathcal{D}^{q} \xrightarrow{\mathbb{P}_{q}} \mathcal{D}^{q+1} \to \cdots$$

There is $Q : L^2(M : E^q) \to \mathcal{D}^{q-1}$ and $R_q : L^2(M, E^q) \to L^2(M, E^q)$ smoothing such that

$$\mathbb{P}_{q-1} Q f = f - R_q f$$

if $f \in \ker \mathbb{P}_q$.

**Proposition**. Let

$$\mathcal{E}^q = \{f \in L^2(M; E^q); \mathbb{P}_{q+1}\mathbb{P}_q f = 0\}.$$

Then

$$\cdots \to \mathcal{E}^{q-1} \xrightarrow{\hat{\mathbb{P}}_{q-1}} \mathcal{E}^{q} \xrightarrow{\hat{\mathbb{P}}_{q}} \mathcal{E}^{q+1} \to \cdots$$

$\hat{\mathbb{P}}_q = \mathbb{P}_q|_{\mathcal{E}^q}$

is a complex. Its cohomology space in degree $q$ is isomorphic to the cohomology of (*), i.e.,

$$\{f \in L^2(M; E^q) : \mathbb{P}_q f \in \operatorname{rg} \mathbb{P}_q \mathbb{P}_{q-1}\} / \operatorname{rg}\left(\mathbb{P}_{q-1} : L^2(M; E^{q-1}) \to L^2(M; E^q)\right)$$

as a vector space.

$M$ closed, $E^q \to M$ Hermitian, $\mathfrak{m}$ is a smooth measure. The sequence is elliptic, $\mathbb{P}_{q+1} \circ \mathbb{P}_q = a_q$ is of order 0
$a_q : E^q \to E^{q+2}$ is a vector bundle

**Lemma**. $\mathcal{D}^q$ is the closure of $C^\infty(M; E^q)$ under the norm defined by

$$(u, v)^2_{\mathbb{P}} = (u, v) + (\mathbb{P}_q u, \mathbb{P}_q v).$$

graph inner product

Proof. Let $\mathcal{D}^q_{\min}$ be the closure of $C^\infty(M; E^q)$ in the graph norm, let $\mathcal{E}^q$ be the subspace of $\mathcal{D}^q$ orthogonal to $\mathcal{D}^q_{\min}$. We will show $\mathcal{E}^q \subset H^1(M; E^q)$. Since $H^1(M; E^q) \subset \mathcal{D}^q_{\min}$, this will show $\mathcal{E}_q = \{0\}$.

From

$$u \in \mathcal{E}^q \iff (\mathbb{P}_q u, \mathbb{P}_q v) = (-u, v) \quad \text{for all } v \in C^\infty(M; E^q).$$

get $\mathbb{P}_q u$ belongs to the domain of the Hilbert space adjoint of

$$\mathbb{P}_q : \mathcal{D}^q \subset L^2(M; E^q) \to L^2(M; E^q)$$

and that as such, $\mathbb{P}^*_q \mathbb{P}_q u = -u$. Thus

$$\Box_q u = \mathbb{P}_{q-1}\mathbb{P}^\star_{q-1} u + \mathbb{P}^\star_q \mathbb{P}_q u = -\mathbb{P}_{q-1}\mathbb{P}^\star_{q-1}\mathbb{P}^\star_q \mathbb{P}_q u - u = -\mathbb{P}_{q-1} a^\star_{q-1}\mathbb{P}_q u - u.$$

But

$$\mathbb{P}_q u \in L^2(M; E^{q+1}) \implies \mathbb{P}_{q-1} a^\star_{q-1} \mathbb{P}_q u \in H^{-1}$$

so $\Box_q u \in H^{-1}(M; E^q)$. Thus $u \in H^1(M; E^q)$, as claimed.

$M$ closed, $E^q \to M$ Hermitian, $\mathfrak{m}$ is a smooth measure. The sequence is elliptic, $\mathbb{P}_{q+1} \circ \mathbb{P}_q = a_q$ is of order 0
$a_q : E^q \to E^{q+2}$ is a vector bundle

## 'Harmonic' sections, Green's operator

Ellipticity gives $\ker \square_q = \mathcal{H}_q$ is finite-dimensional and contained in $C^\infty(M; E)$.

$u \in \ker \square_q \iff (\mathbb{P}_{q-1}u, \mathbb{P}_{q-1}u) + (\mathbb{P}_q u, \mathbb{P}_q u) = 0 \iff \mathbb{P}_q = 0$ & $\mathbb{P}^\star_{q-1}u = 0$.

Let $\Pi^0_q$ be the orthogonal projection on $\mathcal{H}_q$. Then $\Pi^0_q \circ \mathbb{P}_{q-1} = 0$ and $\Pi^0_q \circ \mathbb{P}^\star_q = 0$:

$$(\Pi^0_q \mathbb{P}_{q-1} v, f) = (\mathbb{P}_{q-1} v, \Pi^0_q f) = (v, \mathbb{P}^\star_{q-1}\Pi^0_q f) = 0 \quad \text{if } f \in L^2(M; E^q)$$

Same with $\Pi^0_q \mathbb{P}^\star_q$.

The operator $G_q$ such that

$$\square_q G_q = \mathrm{I} - \Pi^0_q, \quad G_q \square_q = \mathrm{I} - \Pi^0_q$$

is compact and selfadjoint, a pseudodifferential operator of order $-2$.

But $\mathbb{P}_q$ does not 'commute' with $G_q$:

$$\begin{aligned}
\mathbb{P}_q \square_q &= \mathbb{P}_q(\mathbb{P}_{q-1}\mathbb{P}^*_{q-1} + \mathbb{P}^\star_q \mathbb{P}_q) \\
&= a_{q-1}\mathbb{P}^\star_{q-1} + \mathbb{P}_q\mathbb{P}^*_q\mathbb{P}_q + \mathbb{P}^\star_{q+1}\mathbb{P}_{q+1}\mathbb{P}_q - \mathbb{P}^\star_{q+1}\mathbb{P}_{q+1}\mathbb{P}_q \\
&= a_{q-1}\mathbb{P}^\star_{q-1} + \square_{q+1}\mathbb{P}_q - \mathbb{P}^\star_{q+1}a_q
\end{aligned}$$

From this, $\mathbb{P}_q G_q = G_{q+1}\mathbb{P}_q - G_{q+1}(a_{q-1}\mathbb{P}^*_{q-1} - \mathbb{P}^\star_{q+1}a_q)G_q = G_{q+1}\mathbb{P}_q - K^\uparrow_q$

where $K^\uparrow_q$ is a pseudodifferential operator of order $-3$.

$M$ closed, $E^q \to M$ Hermitian, $\mathfrak{m}$ is a smooth measure. The sequence is elliptic, $\mathbb{P}_{q+1} \circ \mathbb{P}_q = a_q$ is of order 0 $a_q : E^q \to E^{q+2}$ is a vector bundle

## 'Projections'

$\Box_q G_q = \Pi_q^0 + (\mathbb{P}_{q-1}\mathbb{P}_{q-1}^* + \mathbb{P}_q^\star \mathbb{P}_q) G_q$

Let $\Pi_q^\uparrow = \mathbb{P}_{q-1}\mathbb{P}_{q-1}^\star G_q$, $\Pi_q^\downarrow = \mathbb{P}_q^\star \mathbb{P}_q G_q$. Then $\mathrm{I} = \Pi_q^0 + \Pi_q^\uparrow + \Pi_q^\downarrow$.
$\Pi_0^\uparrow$ and $\Pi_0^\downarrow$ map into $\mathcal{H}_q^\perp$ and $\Pi_q^\uparrow + \Pi_q^\downarrow : L^2(M; E^q) \to \mathcal{H}_q$ is onto. So the range of $\Pi_q^\uparrow + \Pi_q^\downarrow$ is closed but the individual ranges of $\Pi_q^\uparrow$ and $\Pi_q^\downarrow$ may not. One has

$$\operatorname{rg} \Pi_q^\uparrow \cap \operatorname{rg} \Pi_q^\downarrow = \operatorname{rg} \Pi_q^\uparrow \Pi_q^\downarrow.$$

$\Pi_q^\uparrow \Pi_q^\downarrow = \Pi_q^\downarrow \Pi_q^\uparrow$

**Proposition**. The ranges of $\Pi^\uparrow$ and $\Pi^\downarrow$ are closed iff $\Pi_q^\downarrow \Pi_q^\uparrow$ has closed range, which is the case iff $\operatorname{rg} \Pi_q^\uparrow \cap \operatorname{rg} \Pi_q^\downarrow$ is finite-dimensional. If $\operatorname{rg} \Pi_q^\uparrow \cap \operatorname{rg} \Pi_q^\downarrow$ is finite-dimensional, then it consists of smooth sections.

Finite-dimensionality: $\Pi_q^\uparrow \Pi_q^\downarrow$ is a pseudodifferential operator of order $-2$.

$M$ closed, $E^q \to M$ Hermitian, $\mathfrak{m}$ is a smooth measure. The sequence is elliptic, $\mathbb{P}_{q+1} \circ \mathbb{P}_q = a_q$ is of order 0 $a_q : E^q \to E^{q+2}$ is a vector bundle

Thank you!