REALIZABILITY OF STRICTLY PSEUDOCONVEX CR STRUCTURES IN DIMENSION 5

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1. CR structures in dimension 5

Let M be the unit open ball in $\mathbb{R}^5$. A CR structure on M consists of the datum of a pair of smooth vector fields L_1, L_2 on M with complex coefficients that are in involution,

$$[L_1, L_2] = \lambda_1 L_1 + \lambda_2 L_2$$

for some functions λ_1, λ_2, and such that L_1, L_2, $\overline{L}_1$, $\overline{L}_2$ are pointwise linearly independent. Write $\mathcal{V} = \operatorname{span}\{L_1, L_2\}$; this is a subbundle of $\mathbb{C}TB$.

To describe pseudoconvexity, pick a real vector field T on M which is transverse to $\mathcal{V} \oplus \overline{\mathcal{V}}$. Then $[L_i, \overline{L}_j] = c_{ij} T \mod \mathcal{V} \oplus \overline{\mathcal{V}}$. The Hermitian form $\mathcal{L}$ on $\mathcal{V}$ with entries $[c_{ij}]$ is the Levi form. The CR structure is strictly pseudoconvex if $\mathcal{L}$ is positive definite (or negative definite, but then change the sign of T).

The problem is, to show or to give a counterexample that strictly pseudoconvex CR structure in dimension 5 are realizable. This is an old natural problem.

2. Context: CR structures in dimension $2n + 1$

Let B be the unit open ball in $\mathbb{R}^{2n+1}$, let $L_1, \dots, L_n$ be smooth complex independent vector fields like above, again in involution: $[L_i, L_j] = \sum_k \lambda_{ijk} L_k$. The CR structure is locally realizable if there is a smooth embedding Φ from a neighborhood of 0 in B into $\mathbb{C}^N$ for some N such that $\Phi_*(L_j)$ belongs to $T^{1,0}\mathbb{C}^N$. Such Φ is a "local CR embedding." Use a vector field T transversal to $\mathcal{V} \oplus \overline{\mathcal{V}}$ to define the Levi form $\mathcal{L}$ on the fibers of the subbundle $\operatorname{span}\{L_1, \dots, L_n\} \subset \mathbb{C}TB$. Strict pseudoconvexity again means $\mathcal{L}$ is positive definite.

In dimension 3, Nirenberg [9] showed that there are strictly pseudoconvex CR structures that do not admit local embeddings.

In dimension 5 or greater, Boutet de Monvel [5] proved: If X is a compact CR manifold which is strictly pseudoconvex, then X is globally embeddable into some $\mathbb{C}^N$.

Local embeddability of strictly pseudoconvex CR structures was shown in dimension 7 (Akahori [1]) or higher (Kuranishi [8]) using Nash-Moser. Nirenberg's proof of non-embeddability is a beautiful elementary argument that deforms an embeddable strictly pseudoconvex CR structure (in dimension 3) to one which is not. A particular feature of Nirenberg's case is that the subbundle $\mathcal{V}$ has rank 1 so any deformation is guaranteed to be involutive. Boutet de Monvel uses microlocal analysis, adapted pseudodifferential operators, and an argument similar to Bochner's [4] to produce the embedding (get a sufficiently large family of functions having a desired property, use them as coordinate functions; in this case, functions Z such that $\overline{L}_j Z = 0$).

If one were to try an approach to the problem in dimension 5 along the lines of Nirenberg's, one obstacle to overcome is to maintain involutivity of the deformation. It is an interesting feature of CR structures with non-degenerate Levi forms, that the linearization of the condition that a family of deformations remains involutive can be expressed as $n^3 - n^2$ conditions in $n^2 + n$ unknowns (real valued functions). In dimension 5 ($n = 2$) this gives 4 equations in 6 unknowns, so two degrees of freedom.

Support of the idea that a deformation approach in the style of Nirenberg may work is provided by the underdetermination above and a paper of Jacobowitz and Treves [7], where they use such approach in an argument inspired by Nirenberg's to show that in any dimension, non-degenerate CR structures of Lorenz signature may not be locally embeddable. The existing literature on deformations of CR structures does not seem to help here.

3. Why CR structures matter

The boundary X of a smoothly bounded open set in a complex manifold M acquires a CR structure from the complex structure of the ambient space, as $T^{1,0}M \cap \mathbb{C} \otimes TX$. The holomorphic function theory of the open set is fundamentally determined by the boundary, especially by the Levi form. This is why there is so much work done in CR geometry and related fields (for a start, see [2, 3, 10] and references therein).

4. A remark

Examples of a locally non-embeddable strictly pseudoconvex CR structure $B, \mathcal{V}$ in dimension 5 would imply the existence of a local (in a general sense) CR invariant (different form the Chern-Moser invariants [6]) that expresses the obstruction to $B, \mathcal{V}$ being part of a compact strictly pseudoconvex CR manifold (of dimension 5). This would probably be the most interesting outcome.

References

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