Fredholm condition for isotypical components of invariant operators

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Plan

The problem

Notation:

- G compact Lie group.
- M compact G-manifold (without boundary).
- $E \to M$ a G-equivariant hermitian vector bundle.
- $P : C^\infty(M, E) \to C^\infty(M, E)$ order m pseudodifferential operator.
- We assume P to be G-invariant: $g \circ P = P \circ g$.

Main result (Baldare-Côme-Lesch-V.N.)

A **criterion** for $\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; E)_\alpha$ to be **Fredholm** ($\pi_\alpha(P) =$ restriction to isotypical component corresponding to $\alpha \in \widehat{G}$).

Principal symbols

Main result: in terms of the G-action on the **principal symbol**

$$\sigma_m(P) \in \mathcal{C}^\infty(T^*M \setminus \{0\}; \mathrm{End}(E)).$$

Definition (Principal symbol)

If $P = \sum_{|\alpha|\le m} a_\alpha \partial^\alpha$, then its **principal symbol** is

$$\sigma_m(P)(x,\xi) \ := \ \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha.$$

This definition extends to pseudodifferential operators.

If P is G-invariant then $\sigma_m(P) \in C^\infty(T^*M \setminus \{0\}; \mathrm{End}(E))^G$. ($G$-invariant).

Fredholm operators

Definition

A continuous linear map $T : \mathcal{H} \to \mathcal{H}'$ between Banach spaces is Fredholm if $\ker(T)$ and $\mathrm{coker}(T) := \mathcal{H}'/T(\mathcal{H})$ are finite dimensional. The index:

$$\mathrm{ind}(T) \ := \ \dim\ker(T) - \dim\mathrm{coker}(T).$$

Theorem (Seeley '67, ...)

$P : H^s(M; E) \to H^{s-m}(M; E)$ is Fredholm if, and only if, its principal symbol is invertible (i.e. P is elliptic).

It answers our question if $G = 1$.

Isotypical components

$\widehat{G}$ = isomorphism classes of irreducible unitary representations of G.

Definition

The α-**isotypical component** V_α of V is the largest G-submodule of V that is isomorphic to a multiple of $\alpha \in \widehat{G}$ $(= \sum W,\ \alpha \simeq W \subset V)$.

$$\textbf{Multiplicity} \ = \ m_\alpha(V) \ := \ \dim \mathrm{Hom}_G(\alpha, V) \ = \ \dim \mathrm{Hom}_G(\alpha, V_\alpha)\,.$$

Any finite dimensional G-module V can be decomposed $(\alpha \in \widehat{G})$:

$$V = \bigoplus V_\alpha\,, \text{ and hence}$$

$$[V] \ := \ \sum m_\alpha(V)[\alpha] \ \in \ \{\ \sum k_\alpha[\alpha]\ \} \ =: \ R(G) \ \subset \ \mathcal{C}^\infty(G)^G.$$

$$\text{Notation:} \quad m_\alpha(V) \ =: \ \langle\, [V]\,,\, [\alpha]\,\rangle.$$

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Equivariant index of elliptic operators

Motivation main result (when is $\pi_\alpha(P)$ Fredholm): from index theory.

If $P =$ G-invariant Fredholm, then $[\ker(P)], [\mathrm{coker}(P)] \in R(G)$.

Equivariant index

If P is elliptic and G-invariant, then

$$\mathrm{ind}_G P := [\ker P] - [\mathrm{coker}\, P] \ = \ \sum m_\alpha(P)[\alpha] \in R(G),$$

- Atiyah-Singer index theorem ('68) (for $G = 1$, when $R(G) = \mathbb{Z}$).
- Atiyah-Segal-Singer ('68): $\mathrm{ind}_G(P)(g)$, **value at** $g \in G$.
- Berline-Vergne ('85) : $\mathrm{ind}_G(P)(ge^X)$ in equivariant coh.

Plan

Restriction to isotypical components

Natural Question

To find an explicit formula for the **coefficients** $m_\alpha(P) = \langle\, \mathrm{ind}_G(P)\,,\, [\alpha]\,\rangle$ in $\mathrm{ind}_G(P) = \sum m_\alpha(P)[\alpha]$ (not just the **value** $\mathrm{ind}_G(P)(g)$ at some $g \in G$).

For any $\alpha \in \widehat{G}$ and any G-invariant operator P, let

$$\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; E)_\alpha$$

be obtained by **restriction.** Then

$$\mathrm{ind}_G(P) \ = \ \sum_{\alpha\in\widehat{G}} m_\alpha(P)[\alpha] \ = \ \sum_{\alpha\in\widehat{G}} \frac{\mathrm{ind}(\pi_\alpha(P))}{\dim(\alpha)}[\alpha]\,.$$

Related question

The resulting relation

$$m_\alpha(P) = \frac{\mathrm{ind}(\pi_\alpha(P))}{\dim(\alpha)}.$$

Suggests the following **related question**:

Question

Find **necessary and sufficient cond.** on P for $\pi_\alpha(P)$ to be **Fredholm.**

Also practical application (dimension reduction in numerical methods).

The answer (our main result) will depend on **just the princ symb** $\sigma_m(P)$.

Transverse cotangent space

- $Y_M =$ vector field associated with $Y \in \mathrm{Lie}(G)$.

Definition (The G-transvers space $T^*_G M$)

The G-**transverse cotangent space** $T^*_G M$ is

$$T^*_G M \ := \ \Big\{ \xi \in T^*M \mid \ \langle Y_M, \ \xi \rangle = 0 \ \text{ for all } \ Y \in \mathrm{Lie}(G) \Big\} .$$

Theorem (Atiyah-Singer '73)

If P is invertible on $T^*_G M \setminus \{0\}$, then $(\ker P)_\alpha$ and $(\mathrm{coker}\, P)_\alpha$ are finite dimensional ($\alpha \in \widehat{G}$).

The action of the stabilizers

Let $\xi \in T^*M$ and G_ξ

$$G_\xi \ := \ \{g \in G \mid \ g \cdot \xi = \xi\} \ = \ \textit{stabilizer} \text{ of } \xi \text{ in } G\,.$$

Remark

1. $P =$ pseudodiff. G-invariant $\Rightarrow \sigma_m(P) = G$-invariant;
2. G_ξ acts on E_ξ, and hence $\sigma_m(P)(\xi) \in \mathrm{End}(E_\xi)^{G_\xi}$

$(\gamma \in G_\xi \Rightarrow \gamma(\sigma_m(P)(\xi)) \ = \ \sigma_m(P)(\gamma(\xi)) \ = \ \sigma_m(P)(\xi).)$

Let $m = 0$ and $S_G^*M := T_G^*M \cap \{\|\xi\| = 1\}$ (G-transverse cosphere bundle) and

$$\Xi \ := \ \{\,(\xi, \rho) \in S_G^*M \times \widehat{G}_\xi \ \mid \ (E_\xi)_\rho \neq 0\,\}\,.$$

$E_\xi := E_x$ if $\xi \in T_x^*M$ (we work with the pull-back of E to T^*M).

Decomposition according to isotypic components

The definition $\Xi := \{ (\xi, \rho) \in S^*M \times \widehat{G}_\xi \mid (E_\xi)_\rho \neq 0 \}$ is motivated by the decomposition into isotypical components (for $\xi \in S^*M$):

$$E_\xi = \bigoplus_{\rho \in \widehat{G}_\xi, \rho \subset E_\xi} (E_\xi)_\rho = \bigoplus_{(\xi,\rho)\in\Xi} E_{\xi\rho}$$

and hence

$$\mathrm{End}(E_\xi)^{G_\xi} \simeq \bigoplus_{(\xi,\rho)\in\Xi} \mathrm{End}(E_{\xi\rho})^{G_\xi}.$$

Using this decomposition we obtain a fonction $(m = 0)$

$$\widehat{\sigma}_0(P)(\xi,\rho) := \pi_\rho\big[\sigma_0(P)(\xi,\rho)\big] \,:\, \Xi \to \bigcup_{(\xi,\rho)\in\Xi} \mathrm{End}(E_{\xi\rho})^{G_\xi}\,.$$

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The principal orbit bundle

Theorem (Tom Dieck)

Suppose that M/G is connected. Then, there exists a dense open subset $M_0 \subset M$ such that $\forall x \in M_0$ the stabilizers G_x are minimal and conjugate.

$M_0 =$ **principal orbit bundle.**

For all $x' \in M$ (respectively, for all $\xi' \in T^*M$), there exists $x \in M_0$ such that

$$G_x \subset G_{x'} \qquad \text{(respectivement } G_x \subset G_{\xi'}\text{)}.$$

From now on we assume that: M/G is **connected** and $G_0 \subset G$ is a minimal stabilizer.

α-ellipticity

Recall that : $\Xi := \{(\xi, \rho) \mid \xi \in S^*_G M,\ \rho \in \widehat{G}_\xi,\ E_{\xi\rho} \neq \{0\}\}$.

Notation

Fix $\alpha \in \widehat{G}$ and G_0 = a minimal stabilizer. We let

$$\Xi^\alpha \ := \ \{(\xi, \rho) \in \Xi \mid \mathrm{Hom}(\rho, \alpha)^{G_0} \neq \{0\},\ G_0 \subset G_\xi\}.$$

Theorem (Baldare - Côme - Lesch - N.V.)

Let P be a G-invariant pseudodifferential operator. The restriction $\pi_\alpha(P)$ is Fredholm **iff** *$\sigma_m(P)$ is invertible on Ξ^α (α-elliptic).*

Compare to "P elliptic $\Leftrightarrow$ $\sigma_m(P)$ is invertible on all of Ξ."

Idea of the proof

Equivariant version of Atkinson's theorem:

Theorem

$T \in \mathcal{L}(\mathcal{H}', \mathcal{H})^G$ is Fredholm **iff** *there exists $Q \in \mathcal{L}(\mathcal{H}', \mathcal{H})^G$ such that*

$$TQ - \mathrm{Id} \in \mathcal{K}(\mathcal{H}') \qquad \& \qquad QT - \mathrm{Id} \in \mathcal{K}(\mathcal{H}).$$

Let $\overline{\psi}^m(M)$ be the closure of pseudodiff. operators of order $m \in \{-1, 0\}$. The proof is based on the study of the induced morphism

$$\widetilde{\mathcal{R}} : \mathcal{C}_0(S^*M; \mathrm{End}(E))^G = \overline{\psi}^0(M)^G / \overline{\psi}^{-1}(M)^G \xrightarrow{\pi_\alpha} \mathcal{L}(\mathcal{H}_\alpha)/\mathcal{K}(\mathcal{H}_\alpha)\,.$$

We have that $\pi_\alpha(P)$ is Fredholm if, and only if, $\mathcal{R}(P)$ is invertible.

Idea of the proof (cont.)

It is hence enough to find the kernel of the morphism

$$\mathcal{R}_\alpha : \mathcal{C}_0(S_G^* M; \mathrm{End}(E))^G \to \overline{\psi}^0(M)^G / \overline{\psi}^{-1}(M)^G \to \mathcal{L}(\mathcal{H}_\alpha)/\mathcal{K}(\mathcal{H}_\alpha)\,.$$

obtained from the one introduced on the previous slide.

The reason for introducing Ξ is that we have a canonical identification

$$\Xi/G \simeq \mathrm{Prim}(\mathcal{C}_0(S^* M; \mathrm{End}(E))^G),$$

the primitive ideal spectrum.

This identification associates to a couple $\xi \in S_G^* M$ and $\rho \in \widehat{G}_\xi$, $(E_\xi)_\rho \neq 0$ the kernel of the representation

$$\pi_{\xi,\rho}(P) := \pi_\rho(\sigma_0(P)(\xi))\,.$$

Idea of the proof (cont.)

The kernel of $\mathcal{R}_\alpha$ will correspond to a **closed subset** of Ξ/G, by the definition of the Jacobson (hull-kernel) topology on the primitive ideal spectrum.

Our theorem states that this closed set is the image of

$$\Xi^\alpha \ := \ \{\, (\xi, \rho) \mid \ \xi \in S_G^* M \,, \ G_0 \subset G_\xi \,, \ \rho \in \widehat{G}_\xi \,, \ \mathrm{Hom}(\rho, \alpha)^{G_0} \neq \{0\} \,\}$$

in Ξ/G.

Going further into the details of the proof, there is a natural map

$$\Xi/G \to S_G^* M/G \to M/G \,,$$

which associated to $\ker(\pi_{\xi,\rho})$ the projection of ξ.

This is some sort of "central character map."

To identify the closed subset $\text{Prim}(\ker(\mathcal{R}))^c$ of Ξ/G corresponding to the kernel of the restriction morphism

$$\mathcal{R}_\alpha : \mathcal{C}_0(S_G^*M; \text{End}(E))^G = \overline{\psi}^0(M)^G/\overline{\psi}^{-1}(M)^G \to \mathcal{L}(\mathcal{H}_\alpha)/\mathcal{K}(\mathcal{H}_\alpha),$$

we identify first Ξ_0, its subset that sits above the principal orbit type. Then we show that Ξ_0 is a dense subset of $\text{Prim}(\ker(\mathcal{R}))^c$.

Our results: the multiplicity $m_\alpha(P)$ depends only on $\sigma_m(P)|_{\Xi^\alpha}$.
Question: Formula for $m_\alpha(P)$ using cyclic homology?

Application: Index theory on M/G?

Thank you !