Alle gute Dinge sind drei!

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$u(x,t)$
$$\begin{cases} u_{tt} = (c^2)\, u_{xx} \\ u(0,t) = 0 = u(\ell,t) \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

Sep. vars $T(t)X(x) \to \frac{T''}{T} = \frac{X''}{X} \Rightarrow$ constant Λ

$\to X'' + \lambda X = 0 \qquad \lambda = -\Lambda$

$X(0) = 0 = X(\ell) \longrightarrow$ Solutions are $\underbrace{\sin\left(\frac{n\pi x}{\ell}\right)}_{\text{Eigenfcns. } n\in\mathbb{N}}$

$\lambda_n = \frac{n^2\pi^2}{\ell^2}$. Eigenvalues.

$$u(x,t) = \sum_{n\in\mathbb{N}} \left[\hat{f}_n \cos(n\pi t/\ell) + \hat{g}_n \sin\left(\frac{n\pi t}{\ell}\right)\frac{\ell}{n\pi}\right] \sin\left(\frac{n\pi x}{\ell}\right).$$

More generally $\Delta f + \lambda f = 0$, in $\mathbb{R}^n$, $\Delta = \sum_{i=1}^{n} \partial^2_{x_i}$

$f|_{\text{bdry}} = 0$, on $\Omega \subset \mathbb{R}^n$ bounded.

$\to 0 < \lambda_1 \leq \lambda_2 \ \dots \nearrow \infty$.

$\boxed{Q}$ When are the efcns trig fcns?

Defn A polytope in $\mathbb{R}^1$ is $[a,b]$.
In $\mathbb{R}^n$, bdd. domain Ω s.t. $\partial\Omega = \bigcup_{1}^{m} P_k$
$P_k \cong Q_k$ polytope in $\mathbb{R}^{n-1}$.

Thm (B, N, –, T, V) TFAE for Ω = polytope in $\mathbb{R}^n$.

1. First efcn. extends to $\mathbb{R}$ analytic fcn. on $\mathbb{R}^n$.
2. Ω strictly tesselates $\mathbb{R}^n$.
3. Ω is an alcove.

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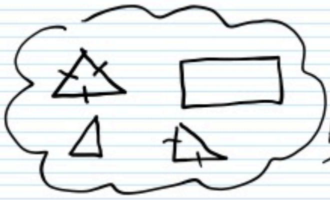
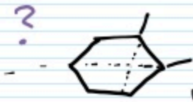
Cor If any of these hold then all efcs are trig.

Strictly tesselates: ① Tesselates by reflection across bdry faces.
② The hyperplanes containing bdry faces never cross the interior of any copy in the tesselation.

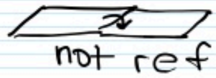
Q What polygons ST $\mathbb{R}^2$?
What don't?



All



not strict.



not refl.

Main Ideas from the proofs

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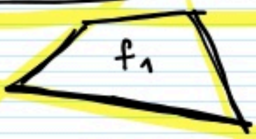
$\underline{1 \Rightarrow 2}$ **Prop (Lamé)** If f_1 $\mathbb{R}$ analytic on $\mathbb{R}^n$,



$f_1 = 0$.

Prop f_1 is odd w.r.t. the bdry faces.

$\partial\Omega$ $r=0$

$\tilde{f}_1(r,z) = -f_1(-r,z)$

$\Delta u + \lambda_1 u = 0$

PDE uniqueness

$\tilde{f}_1 = f_1$.

Thm $f_1 \neq 0$ in Ω.

$\underline{3 \Rightarrow 1}$ Bérard 1980 calculated efcns. of alcoves.

$\underline{2 \Rightarrow 3}$ Root system $= \{v_k\}_{k=1}^{m} = R$ $v_k \neq 0$, span $\mathbb{R}^n$, $\pm v_k \in R$ no other multiples

$v - 2\frac{u \cdot v}{\|u\|^2} u \in R \quad \forall\, u, v \in R$,

$2\frac{u \cdot v}{\|v\|^2} \in \mathbb{Z}$

Let H_v be hyperplane in $\mathbb{R}^n$, $0 \in H_v$, v = normal vector. $H_{v,k} := \{x \in \mathbb{R}^n : v \cdot x = k\}$

A connected component of

$$\mathbb{R}^n \setminus \bigcup_{\substack{v \in R \\ k \in \mathbb{Z}}} H_{v,k}$$

is an alcove.