The Sobolev wavefront set of the causal propagator.

Yafet Sanchez Sanchez
Leibniz Universität Hannover
yess@math.uni-hannover.de
joint work with Elmar Schrohe

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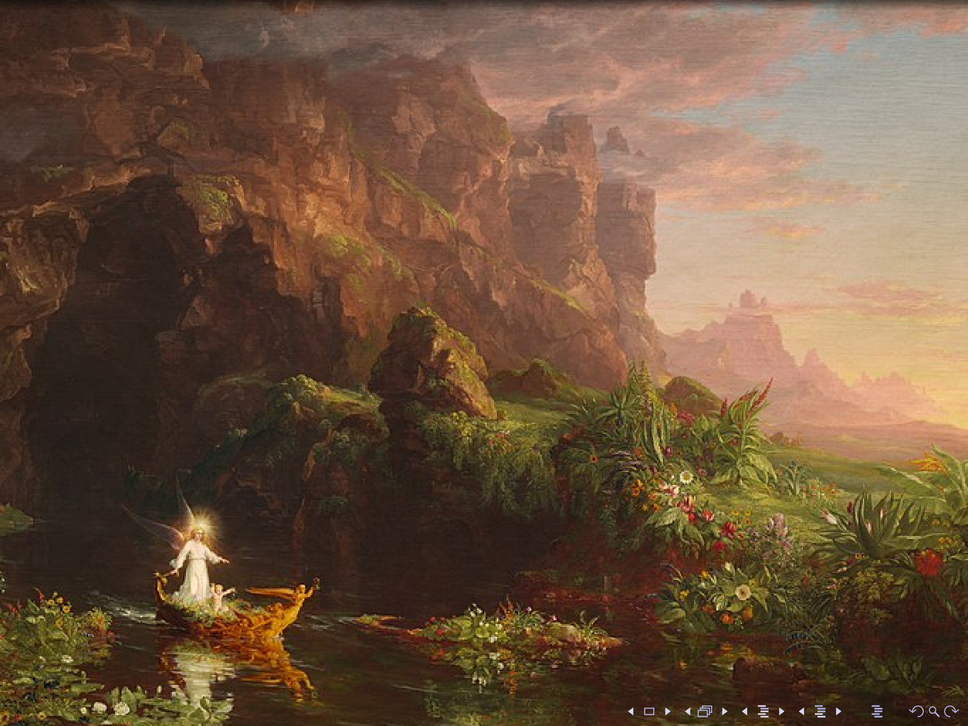


Let

$$\Box u = \partial_{tt} u - \partial_{xx} u = 0$$

on $\mathbb{R}^2$ with initial data

$$u|_{\mathbb{R}} = f(x), \partial_t u|_{\mathbb{R}} = 0$$

and take $f(x)$ smooth except at point x_0.
For which points in $(t, x) \in \mathbb{R}^2$ is u not smooth?
D'Alambert formula:

$$u(t, x) = f(x + t) + f(x - t)$$

Therefore,
$A = \{(t, x) : x = \pm t + x_o\}$

We can refine our analysis by introducing the idea of **Sobolev wavefront set**

A distribution u is microlocally H^s at $(p, \chi) \in (M \times (\mathbb{R}^n \backslash \{0\}))$ if there exists a conic neighbourhood Γ of χ and a smooth function $\varphi \in C_0^\infty(M), \varphi(p) \neq 0$ such that

$$\int_\Gamma (1 + |\chi|^2)^s |\widehat{\varphi u}(\chi)|^2 d^n\chi < \infty.$$

If u is not microlocally H^s at (p, χ), then $(p, \chi) \in WF^s(u)$.

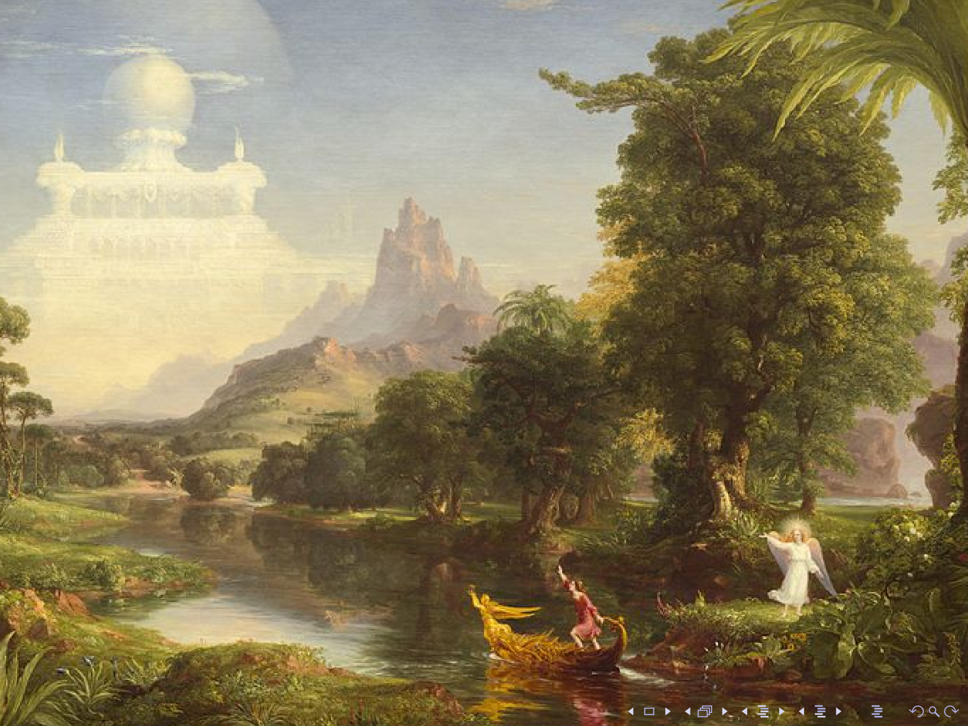
Properties:
for $s \in \mathbb{R}$ $\quad \Pi WF^s(u) \subset \text{singsupp}(u)$; $\quad \overline{\cup_{s\in\mathbb{R}} WF^s(u)} = WF(u)$
Examples:

- $WF^s(u) = \emptyset \iff u \in H^s$
- $\delta_{\mathbb{R}^2}(\phi) = \phi(0); \phi \in D(\mathbb{R}^2)$

$$WF^s(\delta_{\mathbb{R}^2}) = \begin{cases} \emptyset & s < -1 \\ \{0\} \times \mathbb{R}^2 \backslash \{0\} & s \geq -1 \end{cases}$$

- $\psi\delta_{x=y}(\phi) = \int_{\mathbb{R}} \psi(w)\phi(w,w)dw; \phi \in D(\mathbb{R}^2), \psi \in C_c^\infty(\mathbb{R})$
  $WF^s(\psi\delta_{x=y}) =$

$$\begin{cases} \emptyset & s < -\frac{1}{2} \\ \{(w,w) \in \mathbb{R}^2; w \in \text{supp}(\psi)\} \times \{(\xi,-\xi) \in \mathbb{R}^2 \backslash \{0\}\} & s \geq -\frac{1}{2} \end{cases}$$



Ultrastatic setting: $M = \mathbb{R} \times \Sigma$ where Σ is compact manifold and a smooth Lorentzian metric g of the form

$$ds^2 = dt^2 - h_{ij}(x)dx^i dx^j$$

The Klein-Gordon operator is given by

$$P = \partial_{tt} - \Delta_h + m^2$$

The **causal propagator** is given by

$$G := -\frac{\sin(A^{\frac{1}{2}}(t-s))}{A^{\frac{1}{2}}}$$

with Schwartz kernel

$$K_G(t,x;s,y) = -\sum_l \frac{\sin(\lambda_l(t-s))\phi_l(x)\phi_l(y)}{\lambda_l}$$

acting on densities $v\sqrt{h(x)} \otimes u\sqrt{h(y)}$.
Here $A := -\Delta_h \phi + m^2$, the eigenvalues of A are $\{\lambda_j^2\}_{j\in\mathbb{N}}$ and the set of eigenvectors ϕ_l.
This propagator satisfies

$$PG|_{D(M)} = 0 \text{ and } \operatorname{supp}(G(f)) \subset J(\operatorname{supp}(f))$$

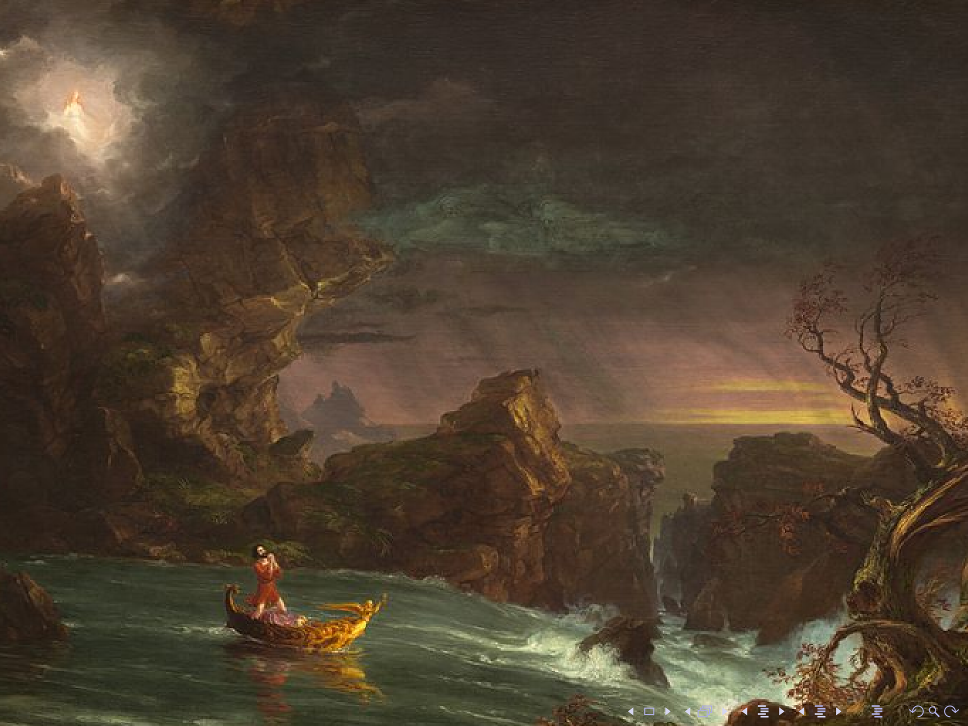
Moreover, we have

$$WF^{s+\frac{3}{2}}(G(f)) \subset WF'(K_G) \circ WF^s(f); \quad WF'(K_G) = C$$

$$WF'(K_G) := \{(p,\eta; q, -\tilde{\eta}) \in T^*(M \times M); (p,\eta; q,\tilde{\eta}) \in WF(K_G)\}$$
$$C = \{(p,\eta,q,\tilde{\eta}) \in T^*(M \times M)\backslash 0;$$
$$g^{ab}(p)\eta_a\eta_b = 0, g^{ab}(q)\tilde{\eta}_a\tilde{\eta}_b = 0, (p,\eta) \sim (q,\tilde{\eta})\}$$

where $(p,\eta) \sim (q,\tilde{\eta})$ means that $\eta, \tilde{\eta}$ are cotangent to the null geodesic γ at p resp. q and parallel transports of each other along γ.

Singularities of solutions propagate along bicharacteristics. The bicharacteristic strips are precisely these geodesics with their cotangent vectors.



C^τ **Ultrastatic setting**: $M = \mathbb{R} \times \Sigma$ where Σ is compact manifold and a C^τ Lorentzian metric g of the form

$$ds^2 = dt^2 - h_{ij}(x)dx^i dx^j$$

The Klein-Gordon operator is given by

$$P = \partial_{tt} - \Delta_h + m^2$$

$$|f|_{C^{0,\alpha}} = \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha},$$
$$\|f\|_{C^{\tau = k+\alpha}} = \|f\|_{C^k} + \max_{|\beta| = k} \left| D^\beta f \right|_{C^{0,\alpha}}$$

In this setting, the solution u of the Cauchy problem
$Pu = 0, u|_\Sigma = g, \partial_t u|_\Sigma = h$ is not smooth, even if g, h are smooth.
In fact, we have

$$WF^{s+\frac{3}{2}}(G(f)) \subset WF'(K_G) \circ WF^s(f) \cup WF_M(K_G)$$

where

$$WF_M(K_G) := \{(p, \eta) \in T^*M; (p, \eta, q, 0) \in WF(K_G) \text{ for some } p \in M\}.$$

Another motivation: The precise structure of the singularities of $WF'(K_G)$ is relevant for quantum field theory.

Theorem (E. Schrohe, Y.S.S)

Let (M, g) *be a* C^τ *ultrastatic spacetime with* $\tau > 2$ *and* K_G *the causal propagator. Then* $WF'^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(K_G) \subset C$ *for every* $\tilde{\epsilon} > 0$

In the smooth case $WF'(K_G) = C$.

Support of K_G

Lemma

Let $(p, q) \in M \times M$ be such that p and q are not causally related, i.e. $p \notin J(q)$. Then $(p, q) \notin supp(K_G)$.

It is enough to show that there are open neighbourhoods U of p and V of q such that K_G vanishes in $W = V \times U$.
To construct the sets U and V we use M is globally hyperbolic ($J(q) \cap (\{t\} \times \Sigma)$ is compact).
Choose smooth functions ψ and ϕ with $\mathrm{supp}(\psi) \subset V$ and $\mathrm{supp}(\phi) \subset U$. Then

$$
\begin{aligned}
K_G(\psi \otimes \phi) = \langle G(\psi), \phi \rangle = \int_M G(\psi)\phi\sqrt{g}dx \\
= \int_{J(\mathrm{supp}(\psi))\cap\mathrm{supp}(\phi)} G(\psi)\phi\sqrt{g}dx \\
= \int_{J(V)\cap U} G(\psi)\phi\sqrt{g}dx = 0.
\end{aligned}
$$

Regularity of K_G

Lemma

$K_G \in H_{loc}^{-\frac{1}{2}-\epsilon}(M \times M)$ for $\epsilon > 0$

Proof.
We show

$$H^{-\frac{1}{2}-\epsilon}(\mathbb{R}^2 \times \Sigma^2)$$
$$= \{u \in \mathcal{S}'(\mathbb{R}^2 \times \Sigma^2);$$
$$\sum_{j,k} \int_{\mathbb{R}^2} (|\xi_0|^2 + |\eta_0|^2 + \lambda_j^2 + \lambda_k^2)^{-\frac{1}{2}-\epsilon)} |\mathcal{F} u_{jk}(\xi_0, \eta_0)|^2 d\xi_0 d\eta_0 < \infty\},$$

with $u_{j,k} = \langle u, \phi_j \otimes \phi_k \rangle \in \mathcal{S}'(\mathbb{R}^2)$. Then,

$$\|\psi K_G\|^2_{H^{-\frac{1}{2}-\epsilon}(M \times M)} = \sum_{j=k} \int_{\mathbb{R}^2} (|\xi_0|^2 + |\eta_0|^2 + \lambda_j^2 + \lambda_k^2)^{(-\frac{1}{2}-\epsilon)}$$
$$\left| \mathcal{F}_{(t,s)\to(\xi_0,\eta_0)} \left(\frac{\psi(t,s)}{\lambda_j} \sin \lambda_j (t-s) \right) (\xi_0, \eta_0) \right|^2 d\xi_0 d\eta_0.$$

We have by direct computation that

$$
\begin{aligned}
&\|\psi K_G\|_{H^{-\frac{1}{2}-\epsilon}(M\times M)} \leq \sum_{j=k} \int_{\mathbb{R}^2} (\lambda_j^2+\lambda_k^2)^{-\frac{1}{2}-\epsilon} \\
&\frac{1}{\lambda_j^2}\left(\frac{C}{(\langle \xi_0-\lambda_j\rangle\langle \eta_0+\lambda_j\rangle)^N}+\frac{C}{(\langle \xi_0+\lambda_j\rangle\langle \eta_0-\lambda_j\rangle)^N}\right)^2 d\xi_0 d\eta_0 \\
&\leq C\sum_{j=k}\frac{(\lambda_j^2+\lambda_k^2)^{-\frac{1}{2}-\epsilon}}{\lambda_j^2} \\
&\leq C\sum_{j}\frac{1}{\lambda_j^{3+2\epsilon}}.
\end{aligned}
$$

Using Weyl's law for non-smooth metrics (Zielinski) we have the estimate $l^{\frac{2}{3}} \leq C\lambda_l^2$ gives

$$
\|\psi K_G\|^2_{H^{-\frac{1}{2}-\epsilon}(M\times M)} \leq \sum_l \frac{C}{\lambda_l^{3+2\epsilon}} \leq \sum_l \frac{C}{l^{1+\epsilon}} < \infty. \qquad (1)
$$

Non-smooth PDO A symbol $p(x,\xi)$ belongs to $C^\tau S^m_{1,\delta}$ if and only if $||D^\alpha_\xi p(x,\xi)||_{C^\tau} \leq C_\alpha \langle\xi\rangle^{m-|\alpha|+\tau\delta}$ and $|D^\alpha_\xi p(x,\xi)| \leq C_\alpha \langle\xi\rangle^{m-|\alpha|}$.
The Pseudodifferential operator $p(x,D)$ associated to the symbol $p(x,\xi) \in C^\tau S^m_{1,\delta}$ is given by

$$p(x,D)u = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)d\xi$$

The operators have the mapping properties $p(x,D) : H^{s+m} \to H^s$ provided $-\tau(1-\delta) < s < \tau$.
Given $p(x,\xi) C^\tau S^m_{1,0}$ there is a decomposition given by

$$p(x,\xi) = p^\#(x,\xi) + p^b(x,\xi)$$

where $p^\#(x,\xi) \in S^m_{1,\delta}$ and $p^b(x,\xi) \in C^\tau S^{m-\tau\delta}_{1,\delta}$ with $\delta \in (0,1)$.

Propagation of singularities.

(Taylor) Let $Q \in C^\tau S^m_{cl}$ with $\tau > 2$ and γ be an integral curve of H_Q. If $v \in D'(X)$ solve $Q^\# v = g$ with $g \in H^\sigma_{mcl}$ on γ and $v \in H^{m-1+\sigma}_{mcl}$ near $q = \gamma(0)$, then $v \in H^{m-1+\sigma}_{mcl}$ on γ.

- Let $Q = P \otimes I$, then $QK_G = 0$ and $Q^\# K_G = Q^b K_G = g \in H^\sigma$
- If $(t, x, s, y, \tilde{\xi}, \tilde{\eta}) \notin WF^{\sigma+1}(K_G)$ using $P \otimes I, I \otimes P$ we have $B(t, x, \tilde{\xi}) \times B(s, y, \tilde{\eta}) \notin WF^{\sigma+1}(K_G)$. The integral curves of $H_{P\otimes I}$ are the bicharacteristics of P in the first set of variables and constant in the second. Similarly with $H_{I\otimes P}$

Since $Q^b(x, D) : H^{s+1-\tau\delta} \to H^{s-1}$ for $(1-\tau)\delta < s < \tau - 1$

Then $\sigma = -\frac{5}{2} + \tau - \tilde{\epsilon}$

This gives propagation of singularities of order

$m - 1 + \sigma = 2 - 1 + \sigma = -\frac{3}{2} + \tau - \tilde{\epsilon}$

Lemma

If $(p, \tilde{\xi}, p, \tilde{\eta}) \in WF^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(K_G)$ *for* $\tilde{\epsilon} > 0$*. Then* $\tilde{\eta} = -\tilde{\xi}$*.*

Let $\tilde{\eta} \neq \lambda\tilde{\xi}$ for $\lambda \in \mathbb{R}$. W choose a Cauchy hypersurface $\Sigma_{t_0} = \{t_0\} \times \Sigma$ such that the null geodesic with initial data $(\tilde{x}, \tilde{\xi})$ and the null geodesic with initial data $(\tilde{x}, \tilde{\eta})$ intersect it. These points of intersections are unique by global hyperbolicity. Moreover, using the condition $\tilde{\eta} \neq \lambda\tilde{\xi}$, we can choose Σ_{t_0} such that these points are distinct. We denote these points by $(t_0, x_0), (t_0, y_0)$. Notice that these points are not causally related. By the propagation of singularities $(t_0, x_0, t_0, y_0) \in \Pi(WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(K_G))$, where Π is the projection from $T^*(M \times M)$ to $M \times M$. However, this is a contradiction since $(t_0, x_0, t_0, y_0) \notin \mathrm{supp}(K_G)$. Therefore, $\tilde{\eta} = \lambda\tilde{\xi}$. Using $(\partial_t + \partial_s)K_G = 0$ we have $\xi^0 = -\eta^0$ which gives $\lambda = -1$ i.e. $\tilde{\eta} = -\tilde{\xi}$.

Theorem (E. Schrohe, Y.S.S)

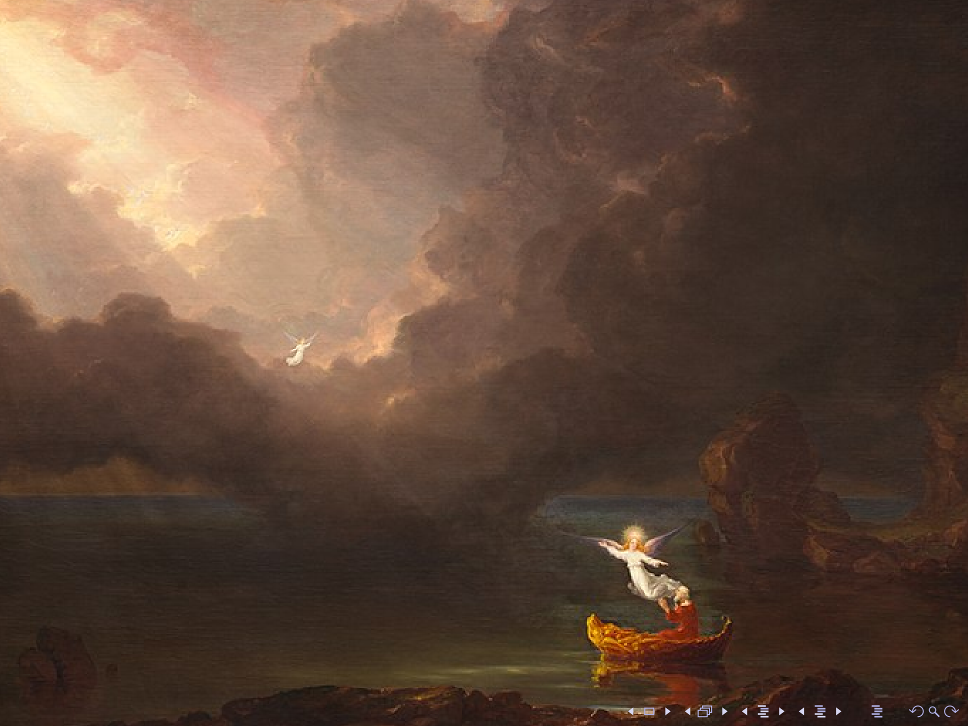
Let (M, g) be a C^τ ultrastatic spacetime with $\tau > 2$ and K_G the causal propagator. Then $WF'^{-\frac{3}{2}+\tau-\tilde{\epsilon}}(K_G) \subset C$ for every $\tilde{\epsilon} > 0$

Let $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in WF^{-\frac{3}{2}-\tilde{\epsilon}+\tau}(K_G)$. The propagation of singularities result gives $(\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, -\tilde{\eta})) \in WF^{-\frac{1}{2}-\tilde{\epsilon}+\tau}(K_G)$. Now we choose a Cauchy surface $\Sigma_{t_1} = \{t_1\} \times \Sigma$ and notice that $(t_1, x_1, \xi_1, t_1, x_1, \tilde{\eta}_1) = (\gamma(\tilde{x}, \tilde{\xi}), \gamma(\tilde{y}, \tilde{\eta})) \cap \Sigma^2$ must satisfy the previous Lemma and therefore is of the form $(t_1, x_1, \xi_1, t_1, x_1, -\tilde{\xi}_1)$. Furthermore, we define the null geodesic $\gamma : (-\infty, \infty) \mapsto M$ as follows

$$\gamma(t) = \begin{cases} \Pi\gamma(\tilde{x}, \tilde{\xi})(t) & t = (-\infty, t_1) \\ \Pi\gamma(\tilde{y}, -\tilde{\eta})(-t) & t = (-t_1, -\infty) \end{cases}$$

where without loss of generality we assumed for $a < t_1 < b$, $\gamma(a) = \gamma(\tilde{x}, \tilde{\xi})(a) = \tilde{x}$ and $\gamma(b) = \gamma(\tilde{y}, -\tilde{\eta})(b) = \tilde{y}$. Moreover $g(\cdot, \dot{\gamma})|_{T_{\tilde{x}}M} = \tilde{\xi}, g(\cdot, \dot{\gamma})|_{T_{\tilde{y}}M} = \tilde{\eta}$.
Therefore, $(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}) \in C' := \{(\tilde{x}, \tilde{\xi}, \tilde{y}, -\tilde{\eta}); (\tilde{x}, \xi; \tilde{y}, \tilde{\eta}) \in C\}$.
This shows

$$WF^{-\frac{3}{2}-\tilde{\epsilon}+\tau}(K_G) \subset C'$$



Future work:

- Time dependent metrics.
- Feynman propagators.

Thank you for your attention.

Artist: Thomas Cole (1801–1848)
Titles: The Voyage of Life: Childhood, The Voyage of Life: Youth, The Voyage of Life: Manhood,The Voyage of Life: Old Age
Date: 1842
Museum: National Gallery of Art
Country: USA