Equivariant Traces for an Algebra of Quantized Canonical Transformations on $\mathbb{R}^n$

Elmar Schrohe
Leibniz Universität Hannover

Microlocal and Global Analysis, Interactions with Geometry
Potsdam, February 21, 2022

Joint work with Anton Savin (RUDN, Moscow)

The Noncommutative Residue

Definition

A trace τ on an algebra $\mathcal{A}$ is a linear map

$$\tau : \mathcal{A} \to \mathbb{C}$$

that vanishes on commutators: $\tau([A, B]) = \tau(AB - BA) = 0$.

- If τ is a trace, then so is $\lambda\tau$, $\lambda \in \mathbb{C}$; the zero map is always a trace.
- When counting traces, we will ignore these.

Example

On the algebra $\mathscr{L}(\mathbb{C}^n)$ there is a unique trace, namely the standard one.

The Noncommutative Residue

Theorem (Wodzicki 1984)

There is a unique trace res on the algebra $\Psi^*_{cl}(M)$ of all classical pseudodifferential operators acting on sections of a vector bundle E over a closed manifold M, given by

$$\operatorname{res} A = \int_{S^*M} \operatorname{tr}_E a_{-n}(x,\xi)\, dS.$$

Here $n = \dim M \geq 2$ and a_{-n} is the component of homogeneity $-n$ in the expansion of the symbol of A.

Remark

- a_{-n} depends on coordinates, the integral over the ξ-sphere does not.
- On $\Psi^0_{cl}(M)$, e.g., there are many more traces (Wodzicki, Lesch-Neira, Paycha-Rosenberg, ...)
- Good reference: Fedosov-Golse-Leichtnam-Schrohe JFA 1996.

The Noncommutative Residue

Applications

- NCR was discovered independently by Guillemin in his ‘soft’ proof of Weyl’s formula
- Plays an important role in noncommutative geometry, as it coincides with Dixmier’s trace on operators of order $-\dim M$ (Connes).
- Derivation of Einstein-Hilbert action in theory of gravitation (Kastler, Kalau-Walze)
- Construction of central extensions in conformal field theory (Khesin and Kravchenko)
- In dimension 1 used by Adler and Manin in work on algebraic aspects of KdV type equations

The Noncommutative Residue

Resolvent Expansions, Zeta Functions and 'Heat' Traces

A arbitrary ψdo, H 'good' reference operator. Then the NCR can be retrieved as the coefficient c_0' resp. $\tilde{c}_0'$ in the following expansions:

- $\operatorname{Tr}(A(H-\lambda)^{-K})$
  $\sim \sum_{j=0}^{\infty} c_j(-\lambda)^{(n+\operatorname{ord} A-j)/\operatorname{ord} H-K} + \sum_{j=0}^{\infty}(c_j' \log(-\lambda) + c_j'')(-\lambda)^{-j-K}$
- $\Gamma(z)\operatorname{Tr}(AH^{-z})$
  $\sim \sum_{j=0}^{\infty} \frac{\tilde{c}_j}{z-(n+\operatorname{ord} A-j)/\operatorname{ord} H} + \sum_{j=0}^{\infty}\left(\frac{\tilde{c}_j'}{(z+j)^2} + \frac{\tilde{c}_j''}{z+j}\right).$
- $\operatorname{Tr}(Ae^{-tH})$
  $\sim \sum_{j=0}^{\infty} \tilde{c}_j t^{(j-n-\operatorname{ord} A)/\operatorname{ord} H} + \sum_{j=0}^{\infty}(\tilde{c}_j' \log t + \tilde{c}_j'')t^j$

Moreover, there is a strong connection between these facts. More later.

Wodzicki, Grubb and Seeley

Shubin Type Pseudodifferential Operators

Now $M = \mathbb{R}^n$.
$\Psi(\mathbb{R}^n)$: all ψdos A whose symbol a satisfies the estimates

$$D_p^\alpha D_x^\beta a(x,p) = (1 + |x| + |p|)^{m-|\alpha|-|\beta|}, \quad m = \operatorname{ord} A.$$

Symbol expansion

$$a(x,p) \sim \sum_{j=0}^{\infty} a_{m-j}(x,p),$$

a_{m-j} homogeneous in (x,p) of degree $m-j$ on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{0\}$. We have an analog of the noncommutative residue for this algebra, namely

$$\operatorname{res} A = \int_{\mathbb{S}^{2n-1}} \operatorname{tr}_E a_{-2n}(x,p) dS.$$

Boggiatto-Nicola

Two Unitary Group Actions on $L^2(\mathbb{R}^n)$

1. Heisenberg-Weyl Operators

$\mathbb{C}^n$ is represented on $L^2(\mathbb{R}^n)$ by Heisenberg-Weyl operators.
For $z = a - ik$,

$$T_z u(x) = e^{ikx - iak/2} u(x-a).$$

Then $T_z T_w = e^{-i\,\mathrm{Im}\langle z,w\rangle/2} T_{z+w}$.

2. Metaplectic Operators

The complex metaplectic group $Mp^c(n) \subseteq \mathscr{L}(L^2(\mathbb{R}^n))$ is generated by all operators

$$S = e^{i(Q+\phi)},$$

where $Q = \mathrm{op}^w(q)$ (Weyl quantization) for a homogeneous real quadratic form $q = q(x,p)$ on $T^*\mathbb{R}^n$, $\phi \in \mathbb{R}$.
$Mp(n)$: subgroup, where $\phi = 0$.

Unitary, since Weyl quantizations of real symbols are self-adjoint.

Metaplectic Operators and Symplectic Matrices

- Let $S = e^{iQ}$, $Q = \mathrm{op}^w(q)$ for a homog. quadratic form $q(x, p)$.
- S defines a (linear) symplectic transformation $\pi(S) \in Sp(T^*\mathbb{R}^n) = Sp(n)$ by evaluating the Hamilton flow defined from q at time 1.
- Yields surjection $\pi : Mp(n) \to Sp(n)$.
- Nontrivial double covering of $Sp(n)$. We can not represent the symplectic matrices by metaplectic operators.
- But we can identify $T^*\mathbb{R}^n \cong \mathbb{C}^n$ via $(x, p) \mapsto p - ix$.
- This identifies $U(n) \subset \mathscr{L}(\mathbb{C}^n)$ with $Sp(n) \cap O(2n) \subset \mathscr{L}(\mathbb{R}^{2n})$.
- We define lift $R : U(n) \to Mp^c(n)$: $U(n)$ is generated by matrices $\exp(B + iA)$ with A a real symmetric $n \times n$ matrix and B a skew-symmetric real $n \times n$ matrix.
- Define $q(x, p) = \frac{1}{2}(x\ p)\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$ and let

$$R\left(e^{B+iA}\right) = e^{-i\,\mathrm{op}^w q}\sqrt{\det(e^{B+iA})} = e^{-i\,\mathrm{op}^w q} e^{i\,\mathrm{tr}\,A/2}.$$

More Details on the Lift $R : U(n) \to Mp^c(n)$

For $U = e^{B+iA}$ with $A = A^t$, $B = -B^t \in \mathscr{L}(\mathbb{R}^n)$ define

$$q(x,p) = \frac{1}{2}(x\ p)\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} \quad \text{and let}$$

$$R\left(e^{B+iA}\right) = e^{-i\,\mathrm{op}^w q}\sqrt{\det(e^{B+iA})} = e^{-i\,\mathrm{op}^w q}e^{i\,\mathrm{tr}\,A/2}.$$

Proposition

The mapping $R : U(n) \longrightarrow Mp^c(n)$, $g \mapsto \pi^{-1}(g)\sqrt{\det g}$, defined near I in $U(n)$, where π^{-1} is the section for $\pi : Mp(n) \to Sp(n)$ with $\pi^{-1}(I) = I$ and $\sqrt{1} = 1$, extends to $U(n)$ as a monomorphism of groups.

A More Explicit Description

$U(n)$ is generated by $O(n)$ and $U(1)$, interpreted as a subgroup of $U(n)$ via $e^{i\varphi} \mapsto \mathrm{diag}(e^{i\varphi}, 1, \ldots, 1)$. We then obtain

$$\begin{aligned} R_g u(x) &= u(g^{-1}(x)), \qquad g \in O(n) \\ R_g u(x) &= e^{i\varphi(\frac{1}{2} - H_1)} u(x), \quad g = \mathrm{diag}(e^{i\varphi}, 1, \ldots, 1), \varphi \neq 0. \end{aligned}$$

where $H_1 = \frac{1}{2}(x_1^2 - \partial_{x_1}^2)$.
In this case, R_g is a fractional Fourier transform. By Mehler's formula

$$R_g u(x) = \sqrt{\frac{1 - i \operatorname{ctg} \varphi}{2\pi}} \int e^{i((x_1^2 + y_1^2)\frac{\operatorname{ctg} \varphi}{2} - \frac{x_1 y_1}{\sin \varphi})} u(y_1, x_2, ..., x_n) dy_1, \quad \varphi \notin \frac{\pi}{2}\mathbb{Z}.$$

Fourier integral operator with quadratic phase.
R_g is the Fourier transform in the first variable for $\varphi = \frac{\pi}{2}$.

The Action on Ψ

$U(n)$ acts on Ψ by conjugation via metaplectic operators
$U(n) \ni g \mapsto (A \mapsto R_g A R_g^{-1})$, $A \in \Psi$.
The principal symbol $\sigma : \Psi \to C(\mathbb{S}^{2n-1})$ satisfies a Egorov theorem

$$\sigma(R_g A R_g^{-1}) = (g^{-1})^* \sigma(A), \quad A \in \Psi.$$

$U(n)$ also acts on the Heisenberg-Weyl operators by conjugation, and

$$R_g T_z R_g^{-1} = T_{gz}.$$

Together we have an action of $\mathbb{C}^n \rtimes U(n)$ on $\Psi(\mathbb{R}^n)$.

Trace Expansions and Equivariant Traces (Work in Progress)

Consider operators

$$D = \sum_j R_{g_j} T_{w_j} A_j,$$

where $g_j \in U(n)$, $w_j \in \mathbb{C}^n$, $A_j \in \Psi(\mathbb{R}^n)$, and the sum is finite.
These operators form an algebra.
Choose an auxiliary operator H in Ψ of order $\mathfrak{m} > 0$ whose principal symbol is positive and scalar, e.g.

$$H = H_0 = \frac{1}{2}(|x|^2 - \Delta).$$

Lemma

Replacing possibly H by $H + c$ for $c > 0$ sufficiently large we will have $H - \lambda$ invertible for all λ in a given sector $S_\vartheta = \{|\arg z| \geq \vartheta\}$, $\vartheta > 0$.

Resolvent Expansion à la Grubb-Seeley

Consider an operator $D = R_g T_w A$. Choose $K \in \mathbb{N}$ such that $-K \operatorname{ord} H + \operatorname{ord} A < -2n$, so that $R_g T_w A(H-\lambda)^{-K}$ is trace class.

Theorem

1. As $\lambda \to \infty$ in the sector it has an expansion (with $m = \dim_{\mathbb{C}}(\mathbb{C}^n)^g$)

$$\operatorname{Tr}(R_g T_w A(H-\lambda)^{-K}) \sim \sum_{j=0}^{\infty} c_j(-\lambda)^{(2m+\operatorname{ord} A - j)/\operatorname{ord} H - K} + \sum_{j=0}^{\infty}(c'_j \log(-\lambda) + c''_j)(-\lambda)^{-j-K}$$

   with suitable coefficients c_j, c'_j, c''_j, $j = 0, 1, \ldots$.

2. c'_0 is independent of the choice of H subject to the above properties.

Resolvent Expansions and Zeta Functions (Grubb-Seeley)

As observed by Grubb and Seeley, the resolvent expansion is the key to the other results. Here is their statement:

For $0 < \delta \leq \pi$ let

$$S_\delta = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda - \pi| < \delta\}$$

$U_r(0)$, $r > 0$ open disk of radius r about the origin in $\mathbb{C}$.
$\mathcal{C}_r$ contour in $\mathbb{C}$ around the boundary of $U_r(0) \cup S_\delta$.
For $0 < \delta < \delta_0 \leq \pi$, $0 < r < r_0$ assume $f : U_{r_0}(0) \cup S_{\delta_0} \to \mathbb{C}$ is holomorphic satisfying $f(\lambda) = O(|\lambda|^{-\alpha})$ for some $\alpha \in]0,1]$ as $|\lambda| \to \infty$ in each sector S_δ.
Define

$$Z(s) = \frac{i}{2\pi} \int_{\mathcal{C}_r} \lambda^{-s} f(\lambda)\, d\lambda, \quad \operatorname{Re} s > 1 - \alpha.$$

Resolvent Expansions and Zeta Functions (Grubb-Seeley)

Proposition

The function $\frac{\pi Z(s)}{\sin \pi s}$ is holomorphic for $\operatorname{Re} s > 1-\alpha$.
Moreover, the following are equivalent:

1. For suitable $\alpha_j \nearrow +\infty$ and $m_j \in \mathbb{N}_0$,
$$f(-\lambda) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} a_{j,l} \lambda^{-\alpha_j} \ln^l \lambda \text{ as } \lambda \to \infty.$$
2. $\frac{\pi Z(s)}{\sin \pi s}$ is meromorphic on $\mathbb{C}$ with the singularity structure
$$\frac{\pi Z(s)}{\sin \pi s} \sim \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} \frac{\alpha_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}},$$
and for each real C_1, C_2 and each $\delta < \delta_0$
$$\left|\frac{\pi Z(s)}{\sin \pi s}\right| \leq Ce^{-\delta|\operatorname{Im} s|}, \quad |\operatorname{Im} s| \geq 1, C_1 \leq \operatorname{Re} s \leq C_2,$$
where C depends on C_1, C_2, and δ.

Zeta Functions

Theorem

1. $z \mapsto \operatorname{Tr}(R_g T_w A H^{-z})$ is holomorphic for $\operatorname{ord} H \operatorname{Re} z > 2n + \operatorname{ord} A$.
2. It extends meromorphically to $\mathbb{C}$ with at most simple poles in $(2m + \operatorname{ord} A - j)/\operatorname{ord} H$.

$$\Gamma(z) \operatorname{Tr}(R_g T_w A H^{-z}) \sim \sum_{j=0}^{\infty} \frac{\tilde{c}_j}{z - (2m + \operatorname{ord} A - j)/\operatorname{ord} H} + \sum_{j=0}^{\infty} \left(\frac{\tilde{c}_j'}{(z+j)^2} + \frac{\tilde{c}_j''}{z+j} \right).$$

3. $\operatorname{Res}_{z=0} \operatorname{Tr}(R_g T_w A H^{-z}) = \tilde{c}_0' = c_0'$.

'Heat' Trace Expansion

Theorem

$t \mapsto \mathrm{Tr}(R_g T_w A \exp(-tH))$ is defined for all $t > 0$.
As $t \to 0^+$ it has an asymptotic expansion

$$\mathrm{Tr}(R_g T_w A e^{-tH}) \sim \sum_{j=0}^{\infty} \tilde{c}_j t^{(j-m-\mathrm{ord}\, A)/\, \mathrm{ord}\, H} + \sum_{j=0}^{\infty} (\tilde{c}_j' \log t + \tilde{c}_j'') t^j$$

with the same coefficients as before.

Noncommutative Residue

Let G be a discrete subgroup of $\mathbb{C}^n \rtimes U(n)$ and

$$D = \sum_j R_{g_j} T_{w_j} A_j.$$

Definition

For $(\bar{w}, \bar{g}) \in G$ we define the noncommutative residue $\mathrm{res}_{\langle(\bar{w},\bar{g})\rangle}$ localized at the conjugacy class $\langle(\bar{w}, \bar{g})\rangle$ in G by

$$\begin{aligned}\mathrm{res}_{\langle(\bar{w},\bar{g})\rangle} D &= \mathrm{ord}\, H \sum_{(w_j,g_j)\in\langle(\bar{w},\bar{g})\rangle} \mathrm{Res}_{z=0} \mathrm{Tr}(R_{g_j} T_{w_j} A_j H^{-z}) \\ &= \mathrm{ord}\, H \sum_{(w_j,g_j)\in\langle(\bar{w},\bar{g})\rangle} c_0'(R_{g_j} T_{w_j} A_j)\end{aligned}$$

with the coefficients c_0' (or $\tilde{c}_0'$) above.

Theorem: Equivariant Traces

$\mathrm{res}_{\langle(\bar{w},\bar{g})\rangle}$ is a trace and can be computed explicitly.

Noncommutative Residue

For diagonal g we can compute the residue explicitly. Let

$$g = \mathrm{diag}(\underbrace{e^{i\varphi_1}, ..., e^{i\varphi_{m_1}}}_{m_1}, \underbrace{i, ..., i}_{m_2}, \underbrace{-i, ..., -i}_{m_3}, \underbrace{-1, ..., -1}_{m_4}, \underbrace{1, ..., 1}_{m_5}) \in U(n), \tag{1}$$

where $\varphi_j \notin \pi\mathbb{Z}/2$ and $m_5 = \dim(\mathbb{C}^n)^g$.

Residue

For $D = R_g T_w A$ with g as above, $w = a - ik \in \mathbb{C}^n$ and $A \in \Psi$

$$\mathrm{Res}_{z=0} \operatorname{Tr}(DH^{-z}) = (2\pi)^{-n+(m_2+m_3)/2} e^{i\langle a,k\rangle/2} \prod_{j=1}^{m_1} \left(\sqrt{i(1-\operatorname{tg}\varphi_j)}\, e^{-\frac{i}{2}k_j^2 \operatorname{tg}\varphi_j} e^{i\delta_j} \right)$$

$$\times \prod_{j=m_1+1}^{m_1+m_2} e^{i\left(\frac{k_j-a_j}{2}\right)^2 + ia_jk_j} \prod_{j=m_1+m_2+1}^{m_1+m_2+m_3} e^{i\left(\frac{k_j+a_j}{2}\right)^2 + ia_jk_j}$$

$$\times \prod_{j=m_1+m_2+m_3+1}^{n-m_5} e^{-ia_jk_j/2} \int\int_{\mathbb{S}^{2m_5-1}} \operatorname{tr}_E \sigma(A)_{-2m_5}(x,\theta)\, dS(\theta) dx.$$

δ_j explicitly given

Noncommutative Residue

If $w = 0$, this simplifies to

$$\operatorname{Res}_{z=0} \operatorname{Tr}(DH^{-z}) = (2\pi)^{-n+(m_2+m_3)/2} \prod_{j=1}^{m_1} \sqrt{i(1-\operatorname{tg}\varphi_j)}$$
$$\times \int \int_{\mathbb{S}^{2m_5-1}} \operatorname{tr}_E \sigma(A)_{-2m_5}(x,\theta)\, dS(\theta) dx.$$

If, additionally, $g = 1$, we obtain the analog of Wodzicki's residue on Ψ.

[Related work on equivariant traces for operator algebras associated with diffeomeorphisms on closed manifold by Dave 2013]

Thank You!

References

1. A. Savin, E. Schrohe, B. Sternin. Elliptic operators associated with groups of quantized canonical transformations. *Bull. Sci. Math.* 155:141-167 (2019)
2. A. Savin, E. Schrohe. Analytic and algebraic indices of elliptic operators associated with discrete groups of quantized canonical transformations. *J. Funct. Anal.* 278, 108400 (2020)
3. A. Savin, E. Schrohe. An Index Formula for Groups of Isometric Linear Canonical Transformations. arXiv:2008.00734
4. A. Savin, E. Schrohe. Local Index Formulae on Noncommutative Orbifolds and Equivariant Zeta Functions for the Affine Metaplectic Group. arXiv:2008.11075