Phase space harmonic analysis of Feynman path integrals

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Plan of the talk

- The elements of Gabor analysis

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- The Gabor analysis of operators
    - The Weyl quantization
    - Metaplectic operators
    - Generalized metaplectic operators

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  - Metaplectic operators
  - Generalized metaplectic operators
- Gabor analysis meets Feynman path integrals
  - Rigorous time-slicing approximations
  - Convergence of integral kernels in the Feynman-Trotter formula
  - Exceptional times?
  - Rates of convergence?

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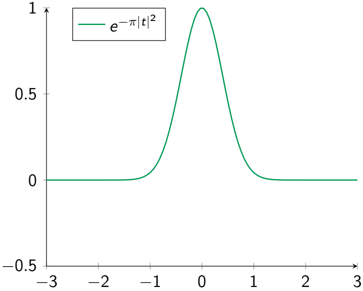

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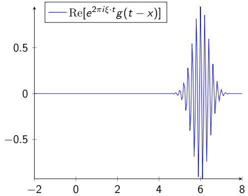
- $\pi(x,\xi) = M_\xi T_x$ is the phase space shift along $(x,\xi) \in \mathbb{R}^{2d}$, with

$$M_\xi f(y) = e^{2\pi i \xi\cdot y} f(y), \quad T_x f(y) = f(y-x), \quad y \in \mathbb{R}^d.$$

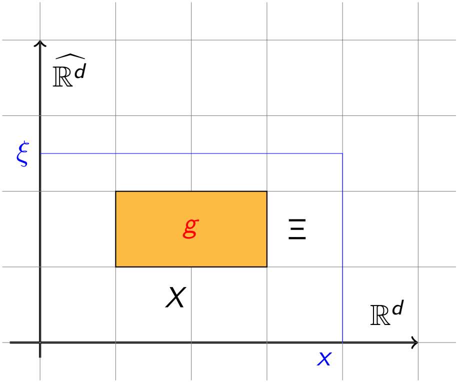


$$(x = 6, \quad \xi = 5)$$

Gabor analysis of functions

Decomposition of a "signal" $f \in \mathcal{S}'(\mathbb{R}^d)$ along Gabor wave packets is known as the Gabor transform (or short-time Fourier transform):

$$V_g f(x,\xi) := \langle f, \pi(x,\xi) g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y)\,\overline{g(y-x)}\,dt, \quad (x,\xi) \in \mathbb{R}^{2d}.$$



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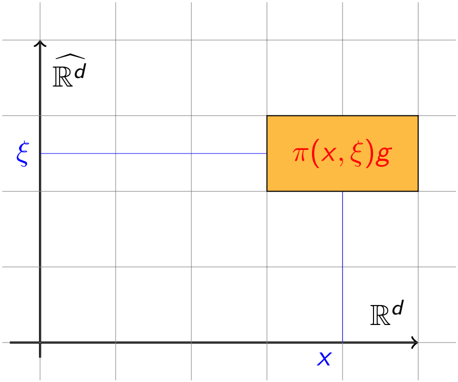

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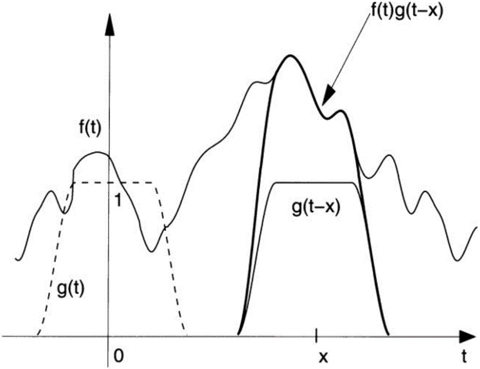

Modulation spaces

For $1 \leq p \leq \infty$ we introduce the modulation spaces

$$M^p(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^p} < \infty \right\},$$

where $\|f\|_{M^p(\mathbb{R}^d)} := \|V_g f\|_{L^p(\mathbb{R}^{2d})}$.

M^p is a Banach space whose definition is independent of the window: if $g, \gamma \in \mathcal{S} \setminus \{0\}$ then $\|V_g f\|_{L^p}$ and $\|V_\gamma f\|_{L^p}$ are equivalent norms for $f \in M^p$.

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Moreover, if $p < \infty$ then $(M^p)' \simeq M^{p'}$, with $1/p + 1/p' = 1$.

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A Banach-Gelfand triple for Gabor analysis:

$$(\mathcal{S} \hookrightarrow) \qquad M^1 \quad \subset \quad M^2 \simeq L^2 \quad \subset \quad M^\infty \qquad (\subset \mathcal{S}').$$

Gabor analysis of operators

Consider a linear continuous operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and $g, \gamma \in \mathcal{S}$ with $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$.

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Several characterizations are known for the Gabor kernels of pseudodifferential operators, Fourier integral operators and Schrödinger-type propagators.

The Weyl quantization and the Wigner distribution

The pseudodifferential operator with Weyl symbol $\sigma : \mathbb{R}^{2d} \to \mathbb{C}$:

$$\mathrm{op_w}(\sigma)f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

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where we introduced the Wigner-Ville distribution ('32, '48)

$$W(f,g)(x,\xi) \coloneqq \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot y} f\left(x+\frac{y}{2}\right)\overline{g\left(x-\frac{y}{2}\right)}dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

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A bridge to Gabor analysis:

$$f \in M^p(\mathbb{R}^d) \quad \Longleftrightarrow \quad W(f,g) \in L^p(\mathbb{R}^{2d}) \quad \forall g \in \mathcal{S}(\mathbb{R}^d).$$

Modulation spaces as symbol classes

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Hints on regularity and inclusions - for $s > d$:

$$M^{\infty}_{0,s}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d) \subset (\mathcal{F}L^1(\mathbb{R}^d))_{\mathrm{loc}} \cap L^{\infty}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d).$$

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- $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ *if and only if there exists a function $H \in L^1(\mathbb{R}^{2d})$ such that*

$$|\langle \sigma^{\mathrm{w}} \pi(z) g, \pi(w)\gamma \rangle| \le H(w - z), \quad z, w \in \mathbb{R}^{2d}.$$

Exploiting sparsity

Theorem (Sjöstrand '94, Gröchenig '06)

Let X be any of the spaces $M^{\infty}_{0,s}(\mathbb{R}^{2d})$, $s > 2d$, or $M^{\infty,1}(\mathbb{R}^{2d})$.

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3. Exploit the structure of the Gabor matrix. □

Metaplectic operators - take one

Let $S \in \mathrm{Sp}(d,\mathbb{R})$ be a symplectic matrix, that is

$$S^\top J S = J, \qquad J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

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Let $S \in \mathrm{Sp}(d,\mathbb{R})$ be a symplectic matrix, that is

$$S^\top J S = J, \qquad J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

There exists a double-valued unitary representation μ of $\mathrm{Sp}(d,\mathbb{R})$ on $L^2(\mathbb{R}^d)$, called the metaplectic representation, such that the metaplectic operator $\mu(S)$ satisfies the intertwining relation

$$\pi'(Sz) = \mu(S)\pi'(z)\mu(S)^{-1}, \quad z \in \mathbb{R}^{2d},$$

where we introduced the symmetric time-frequency shift

$$\pi'(x,\xi) = M_{\frac{\xi}{2}} T_x M_{\frac{\xi}{2}} = e^{-\pi i x\cdot\xi}\pi(x,\xi).$$

Metaplectic operators and Schrödinger propagators

Let Q be a real quadratic form on $\mathbb{R}^{2d}$, namely

$$Q(x,\xi) = \frac{1}{2}A\xi \cdot \xi + Bx \cdot \xi + \frac{1}{2}Cx \cdot x, \quad A, C \in \mathbb{R}^{d\times d}_{\mathrm{sym}},$$

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and consider the Schrödinger equation with Hamiltonian $H_0 = Q^{\mathrm{w}}$ (Weyl quantization) and $V = 0$, where

$$Q^{\mathrm{w}} = -\frac{1}{8\pi^2}\sum_{j,k=1}^{d} A_{j,k}\partial^2_{j,k} - \frac{i}{2\pi}\sum_{j,k=1}^{d} B_{j,k}x_j\partial_k - \frac{i}{4\pi}\mathrm{Tr}(B) + \frac{1}{2}\sum_{j,k=1}^{d} C_{j,k}x_jx_k.$$

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The propagator $U_0(t) = e^{-2\pi i t Q^{\mathrm{w}}}$ is a metaplectic operator, that is

$$U_0(t) = e^{-2\pi i t Q^{\mathrm{w}}} = c(t)\mu(S_t),$$

where $t \mapsto S_t \in \mathrm{Sp}(d,\mathbb{R})$ is the solution of the classical equations of motion with Hamiltonian $Q(x,\xi)$ in phase space.

Gabor analysis of metaplectic operators

Theorem (The Gabor matrix of $\mu(S)$ is sparse)

Consider $U_0(t) = \mu(S_t)$ *as before and* $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$*. For any* $t \in \mathbb{R}$*,* $N \in \mathbb{N}$ *there exists* $C_{t,N} > 0$ *such that*

$$|\langle U_0(t)\pi(z)g, \pi(w)\gamma\rangle| \leq C(1 + |w - S_t z|)^{-N}, \quad w, z \in \mathbb{R}^{2d}.$$

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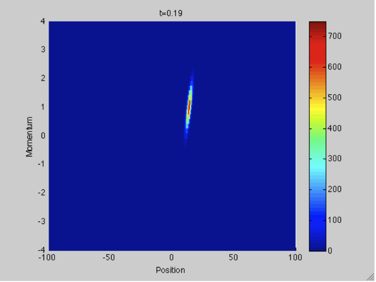

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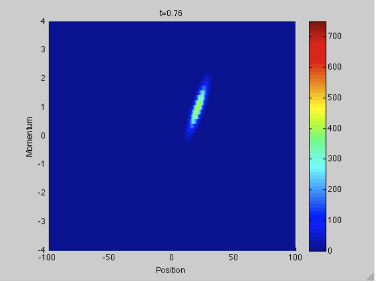

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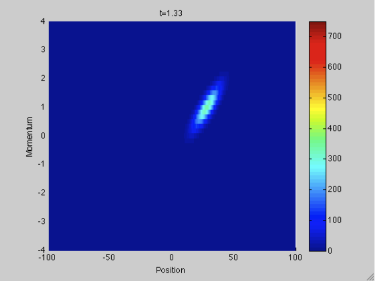

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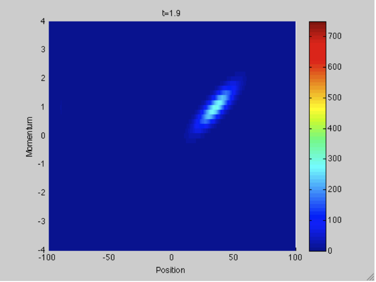

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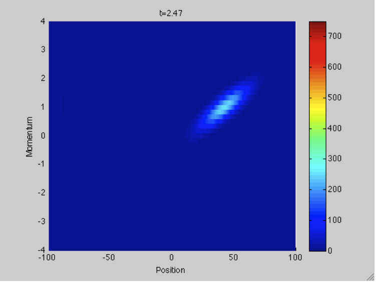

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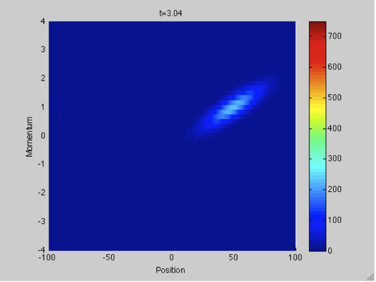

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Theorem (Boundedness on modulation spaces)

Any metaplectic operator $\mu(S)$ *is bounded on* $M^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$.

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Theorem (Cordero, Nicola, T. - ACHA 2021)

Consider $1 \leq p \leq \infty$. There exists $C > 0$ such that, for any $f \in M^p(\mathbb{R}^d)$ and $S \in \mathrm{Sp}(d, \mathbb{R})$,

$$\|\mu(S)f\|_{M^p} \leq C(\sigma_1\sigma_2 \cdots \sigma_d)^{|1/2-1/p|}\|f\|_{M^p},$$

where $\sigma_1 \geq \ldots \geq \sigma_d \geq 1$ are the largest singular values of S.

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$\mathfrak{E}$ is the zero set of an analytic function, therefore:

- $\mathfrak{E} = \mathbb{R}$ ($H_0 = 0$) or $\mathfrak{E}$ is a discrete, hence at most countable, subset of $\mathbb{R}$;
- $\mathfrak{E}$ is non-empty: we always have $0 \in \mathfrak{E}$;
- $\mathfrak{E}$ may be the smallest possible: $H_0 = -\Delta/2 \Longrightarrow \mathfrak{E} = \{0\}$.

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"Almost every symplectic matrix is free"

The subset of free symplectic matrices has codimension 1 in $\mathrm{Sp}(d, \mathbb{R})$.

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Theorem (Quadratic Fourier transforms)

For any $t \in \mathbb{R} \setminus \mathfrak{E}$ and $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\mu^\hbar(S_t)f(x) = c(t)(2\pi\hbar)^{-d/2}|\det B_t|^{-1} \int_{\mathbb{R}^d} e^{\frac{i}{\hbar}\Phi_t(x,y)} f(y)dy,$$

$$\Phi_t(x,y) = \frac{1}{2} D_t B_t^{-1} x \cdot x - B_t^{-1} x \cdot y + \frac{1}{2} B_t^{-1} A_t y \cdot y, \quad c(t) \in \mathbb{C}, \, |c(t)| = 1.$$

More on exceptional times

The harmonic oscillator

$$Q(x,\xi) = \frac{1}{2}|\xi|^2 + \frac{1}{2}|x|^2 \quad \rightsquigarrow \quad H_0 = Q^{\mathrm{w}} = -\frac{1}{8\pi^2}\Delta + \frac{1}{2}|x|^2$$

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For $k \in \mathbb{Z}$ the **Mehler kernel** is - up to phase factors:

$$u_t(x,y) \asymp \begin{cases} |\sin t|^{-d/2} \exp\left(\pi i \frac{x^2+y^2}{\tan t} - 2\pi i \frac{x\cdot y}{\sin t}\right) & (\pi k < t < \pi(k+1)) \\ \delta((-1)^k x - y) & (t = k\pi) \end{cases}.$$

Generalized metaplectic operators

Combining metaplectic and Weyl operators like in $T = \mu(S)\sigma^{\mathrm{w}}$ or $T = \rho^{\mathrm{w}}\mu(S)$ with $\sigma, \rho \in M^{\infty,1}(\mathbb{R}^{2d})$ we obtain the so-called generalized metaplectic operators associated with S - we write $T \in FIO(S)$.

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$$Tf(x) = c(t) \int_{\mathbb{R}^d} e^{2\pi i \Phi_t(x,y)} a_t(x,y) f(y) dy,$$

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  now with an extra amplitude $a_t(x,y) \in M^{\infty,1}(\mathbb{R}^{2d})$.
- T is still well localized in phase space near the graph of S_t:

$$|\langle T(t)\pi(z)g, \pi(w)g\rangle| \le H_t(w - S_t z), \text{ for some } H_t \in L^1(\mathbb{R}^{2d}).$$

Feynman's formulation of quantum mechanics

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Assumption (Feynman '48)

The integral kernel $u_t(x,y)$ of any Schrödinger propagator $U(t) = e^{-\frac{i}{\hbar}tH}$ with quantum Hamiltonian H can be represented as a ***path integral****:*

$$u_t(x,y) = \int e^{\frac{i}{\hbar}S[\gamma]}\mathcal{D}\gamma,$$

where $S[\gamma] = \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), \tau)d\tau$ is the ***action functional*** *(L being the classical Lagrangian) corresponding to a path γ satisfying $\gamma(0) = y$ and $\gamma(t) = x$.*

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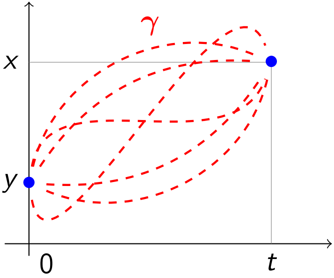

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- The **Trotter product formula**: under suitable assumptions on the potential V we have

$$U(t)f = e^{-\frac{i}{\hbar}t(H_0+V)}f = \lim_{n\to\infty} E_n(t)f, \quad f \in L^2(\mathbb{R}^d),$$

  where the **Feynman-Trotter approximate propagators** are

$$E_n(t) = \left(e^{-\frac{i}{\hbar}\frac{t}{n}H_0}e^{-\frac{i}{\hbar}\frac{t}{n}V}\right)^n.$$

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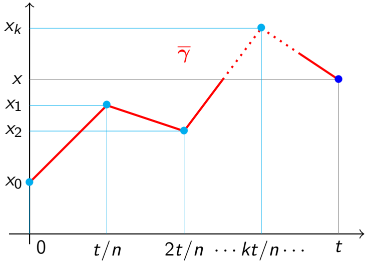

$$S_n(t;x_0,\ldots,x_{n-1},x) \approx S[\overline{\gamma}]$$

Kernels or operators?

... the limit exists and we may write

$$K(b,a) = \lim_{\epsilon\to 0} \frac{1}{A} \int \cdots \int \int e^{\frac{i}{\hbar} S[b,a]} \frac{dx_1}{A} \frac{dx_2}{A} \cdots \frac{dx_{N-1}}{A} \qquad (\bigstar)$$

[where $A = (2\pi i\hbar\epsilon/m)^{1/2}$ and $N\epsilon = t_b - t_a$] ...

... we shall write the sum over all paths in a less restrictive notation as

$$K(b,a) = \int_{t_a}^{t_b} e^{\frac{i}{\hbar} S[b,a]} \mathcal{D}x(t) \qquad (\bigstar)$$

which we shall call a path integral.

— *Feynman and Hibbs, Quantum Mechanics and Path Integrals (1965)*

Kernels or operators?

> … The Trotter product formula shows that the transition from (★) to (★) can be made rigorously on the level of operators rather than integral kernels, under suitable conditions on the potential V…
>
> — *Folland, Quantum Field Theory - A Tourist Guide for Mathematicians (2008)*

Partial results on convergence at the level of integral kernels were proved by **Fujiwara** ('79, '80) for different time-slicing approximations (involving oscillatory integral operators).

Feynman was right!

Set $\hbar = 1/2\pi$ for consistency. Our problem:

$$\begin{cases} i\partial_t \psi = 2\pi H_0 \psi \\ \psi(0,x) = f(x), \end{cases}$$

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where

- H_0 is the Weyl quantization of a real quadratic form $Q(x,\xi)$ on $\mathbb{R}^{2d}$, and
- V is a Weyl operator with symbol σ in any of the following spaces:

$$C_b^\infty(\mathbb{R}^{2d}) \quad (\subset) \quad M^\infty_{0,s}(\mathbb{R}^{2d}),\ s > 2d, \quad (\subset) \quad M^{\infty,1}(\mathbb{R}^{2d})$$

where

$$C_b^\infty(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : |\partial^\alpha f| \le C_\alpha, \quad \forall \alpha \in \mathbb{N}^d\} \ (= S^0_{0,0}).$$

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The Trotter product formula holds in this setting

$$e^{-2\pi i t(H_0+V)} f = \lim_{n\to\infty} E_n(t) f, \qquad E_n(t) = \left(e^{-2\pi i \frac{t}{n} H_0} e^{-2\pi i \frac{t}{n} V} \right)^n.$$

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Simple examples of potentials

Multiplication by V:

$$V(x)f(x) = (V \otimes 1)^{\mathrm{w}} f(x).$$

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A well-known class of potentials

$V \in \mathcal{FM}(\mathbb{R}^d)$ is the Fourier transform of a complex (finite) measure (Itô ('61, '67), Albeverio & Mazzucchi (since '70), Smolyanov).

$$\mathcal{FM}(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d).$$

Potentials with $M^{\infty}_{0,s}$ regularity

Let $e_{n,t}(x,y)$ be the distribution kernel of $E_n(t)$ and $u_t(x,y)$ that of $U(t)$.

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- *$e_{n,t} \to u_t$ in $(\mathcal{F}L^1_r(\mathbb{R}^{2d}))_{\mathrm{loc}}$ for any $0 < r < s - 2d$, hence uniformly on compact subsets.*

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Two characterizations

$$C_b^\infty(\mathbb{R}^{2d}) = \bigcap_{s \geq 0} M_{0,s}^\infty(\mathbb{R}^{2d}), \quad C^\infty(\mathbb{R}^{2d}) = \bigcap_{r > 0} (\mathcal{F} L_r^1(\mathbb{R}^{2d}))_{\mathrm{loc}}.$$

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Let $H_0 = Q^{\mathrm{w}}$ *and* $V = \sigma^{\mathrm{w}}$ *as above with* $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$.

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A useful splitting lemma

For any $\epsilon > 0$ and $f \in M^{\infty,1}(\mathbb{R}^d)$, there exist $f_1 \in C_b^{\infty}(\mathbb{R}^d)$ and $f_2 \in M^{\infty,1}(\mathbb{R}^d)$ such that

$$f = f_1 + f_2, \qquad \|f_2\|_{M^{\infty,1}} \le \epsilon.$$

Ingredients of the proof

Step 1. The Schrödinger propagator $U(t) = e^{-2\pi i t(H_0+V)}$ is a generalized metaplectic operator:

$$U(t) = U_0(t)\alpha_t^{\mathrm{w}} \in FIO(S_t),$$

$$\alpha_t = 1 + \sum_{n\geq 1}(-2\pi i)^n \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} \prod_{k=1}^{n} (\sigma \circ S_{\tau_k})\, d\tau_n \cdots d\tau_1 \in M^{\infty,1}(\mathbb{R}^{2d}).$$

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Step 2. The Trotter parametrices are generalized metaplectic operators as well!

$$E_n(t) = U_0(t)\rho_{t,n}^{\mathrm{w}} \in FIO(S_t),$$

for suitable $\rho_{n,t}$ in bounded subsets of $M^{\infty,1}$ (uniformly w.r.t. n), precisely

$$\rho_{n,t} = \prod_{k=0}^{n-1}\left(1 + 2\pi i \frac{t}{n}\left(\rho_0 \circ S_{k\frac{t}{n}}\right)\right) \in M^{\infty,1}(\mathbb{R}^{2d}).$$

Step 3. Generalized metaplectic operators have integral representations as quadratic Fourier transforms for non-exceptional times:

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Step 6. Transfer the convergence result to $e_{n,t} \to u_t$ via the Gabor transform.

A not so wild behaviour at exceptional times

Theorem (Feichtinger, Nicola, T. - Springer INDAM 2021)

Let $H_0 = Q^{\mathrm{w}}$ as above and $V = \sigma^{\mathrm{w}}$ with $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$. For any $t \in \mathbb{R}$:

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$$M^{\infty}(\mathbb{R}^{2d}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|V_g f\|_{L^\infty(\mathbb{R}^{2d})} < \infty,\ g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \right\}.$$

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- $e_{n,t} \to u_t$ *in the weak-* topology on* $M^{\infty}(\mathbb{R}^{2d})$, *that is*

$$\langle e_{n,t}, \phi \rangle \to \langle u_t, \phi \rangle, \quad \phi \in M^1(\mathbb{R}^d).$$

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There are parametrices $\widetilde{E_n}(t) = U_0(t)\tilde{\rho}^{\mathrm{w}}_{n,t} \in FIO(S_t)$ such that

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for any compact subset $K \subset \mathbb{R}^{2d}$ and a bump function Ψ_K on K,

$$\sup_{z \in K} |u_t(z) - \widetilde{e_{n,t}}(z)| \lesssim_{\Psi_K} \frac{C'(t)}{n}.$$

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