An Horizontal Chern-Gauss-Bonnet formula on totally-geodesic foliations

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Classical Chern-Gauss-Bonnet Theorem

The classical Chern-Gauss-Bonnet theorem gives a relationship between the Euler characteristic of a manifold and local curvature quantities:

Theorem

Let $(\mathbb{M}, g)$ *be a compact, even-dimensional, orientable Riemannian manifold without boundary and let* Ω *be the curvature form of some metric connection* ∇. *Then*

$$\int_{\mathbb{M}} \frac{1}{(2\pi)^n} \det(\Omega)^{1/2} = \chi(\mathbb{M})$$

Totally-geodesic foliations

Let $(\mathbb{M}, g)$ be a smooth $n+m$ dimensional Riemannian manifold equipped with a foliation with m-dimensional leaves.

- We call the subbundle $\mathcal{V}$ tangent to the leaves vertical.
- Let $\mathcal{H}$ be a transverse, bracket-generating subbundle (called horizontal).

vector fields $X \in \Gamma(\mathcal{H}), Z \in \Gamma(\mathcal{V})$, we insist that the foliation is

- Riemannian:
$$(\mathcal{L}_Z g)(X, X) = 0$$
- Totally-geodesic:
$$(\mathcal{L}_X g)(Z, Z) = 0$$

Bott Connection

There exists a unique metric connection ∇ on $(\mathbb{M}, g)$ such that

1. $\mathcal{H}$ and $\mathcal{V}$ are ∇-parallel,
2. The torsion T of ∇ satisfies
   - $T(\mathcal{H}, \mathcal{H}) \subset \mathcal{V}$,
   - $T(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$
3. For every $X, Y \in \Gamma(\mathcal{H}), Z, V \in \Gamma(\mathcal{V})$,
   - $\langle T(X, Z), Y \rangle_{\mathcal{H}} = \langle T(Y, Z), X \rangle_{\mathcal{H}}$
   - $\langle T(Z, X), V \rangle_{\mathcal{V}} = \langle T(V, X), Z \rangle_{\mathcal{V}}$.

This is called the Bott connection.

Hladky-Bott Connection

We can determine ∇ using the Levi-Civita connection ∇^g as

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{H}}[X, Y] & Y \in \Gamma(\mathcal{H}), X \in \Gamma(\mathcal{V}) \\ \pi_{\mathcal{V}}[X, Y] & Y \in \Gamma(\mathcal{V}), X \in \Gamma(\mathcal{H}) \\ \pi_{\mathcal{V}} \nabla^g_X Y & X, Y \in \Gamma(\mathcal{V}) \end{cases}$$

which has torsion

$$T(X, Y) = -\pi_{\mathcal{V}}[\pi_{\mathcal{H}} X, \pi_{\mathcal{H}} Y]$$

J Map

We can associate to each vector field $Z \in \Gamma(T\mathbb{M})$ an endomorphism J_Z of $T\mathbb{M}$ defined by

$$\langle J_Z X, Y \rangle = \langle Z, T(X, Y) \rangle$$

which can be viewed as

$$J \colon \mathcal{V} \to \mathbf{End}(\mathcal{H}), \quad Z \mapsto J_Z$$

Canonical variation metric

We consider the family of metrics parameterized by $\varepsilon > 0$

$$g^\varepsilon = g_\mathcal{H} \oplus \frac{1}{\varepsilon} g_\mathcal{V}$$

and observe:

- If $(\mathbb{M}, g)$ is a totally-geodesic, Riemannian foliation, so is $(\mathbb{M}, g^\varepsilon)$.
- The connection

  $$\hat{\nabla}^\varepsilon_X Y := \nabla_X Y + \frac{1}{\varepsilon} J_X Y$$

  is metric, preserves $\mathcal{H}, \mathcal{V}$, and is also $g^{\varepsilon'}$-metric for any $\varepsilon' > 0$.
- The adjoint connection

  $$\nabla^\varepsilon_X Y := \hat{\nabla}^\varepsilon_X Y - \hat{T}^\varepsilon(X, Y)$$

  is also metric.

Horizontal Laplacian on functions

We now introduce our notion of horizontal Laplacian. On functions $f \in C^\infty(\mathbb{M})$, we define

$$\Delta_{\mathcal{H}} f = \mathrm{tr}_{\mathcal{H}} \nabla_\times df(\times)$$

- $\mathcal{H}$ bracket generating and Hormander's condition imply that $\Delta_{\mathcal{H}}$ is hypoelliptic.
- Since $\mathcal{H}$ and $g_{\mathcal{H}}$ are independent of ε (with respect to the metric, g^ε), so is $\Delta_{\mathcal{H}}$.

Horizontal Laplacian on forms

We desire a horizontal Laplacian $\Delta_{\mathcal{H},\varepsilon}$ acting on differential forms satisfying the following:

- $\Delta_{\mathcal{H},\varepsilon} f = \Delta_{\mathcal{H}} f$ for smooth functions f.
- Weitzenböck type:

$$\Delta_{\mathcal{H},\varepsilon} = \mathrm{tr}_{\mathcal{H}} \tilde{\nabla}^2_{\times,\times} + \mathcal{R}_\varepsilon$$

  with $\tilde{\nabla}$ some g^ε-metric connection and $\mathcal{R}_\varepsilon$ a zeroth-order operator.
- Commutativity with exterior derivative:

$$[\Delta_{\mathcal{H},\varepsilon}, d] = 0$$

Horizontal Laplacian on forms

Following (Baudoin, Grong 2019) these conditions uniquely determine $\Delta_{\mathcal{H},\varepsilon}$, which can be discribed concretely by

$$\Delta_{\mathcal{H},\varepsilon} = -\delta_{\mathcal{H},\varepsilon} d - d\delta_{\mathcal{H},\varepsilon}$$

with the horizontal divergence operator acts on smooth forms η by

$$\delta_{\mathcal{H},\varepsilon}\eta = -\operatorname{tr}_{\mathcal{H}}(\nabla^{\varepsilon}_{\times}\eta)(\times,\cdot)$$

Symmetry of the Horizontal Laplacian

Proposition

$\Delta_{\mathcal{H},\varepsilon}$ is symmetric for the L^2-inner product for g^ε if and only if

$$(\nabla_v J)_w = -\frac{1}{2\varepsilon}[J_v, J_w]$$

Note in particular that in this case,

$$\nabla_{\mathcal{H}} J = 0.$$

Consequences of symmetry

Supposing $\Delta_{\mathcal{H},\varepsilon}$ is symmetric for some $\varepsilon > 0$, it follows that

- $\mathcal{H}$ has step two. That is, $\mathcal{H}^2 := \mathcal{H} + [\mathcal{H}, \mathcal{H}] = T\mathbb{M}$.
- T is surjective on $\mathcal{V}$.
- The tangent cones of the metric space $(\mathbb{M}, d_{cc})$ are all isometric.

McKean-Singer theorem

We will assume herein that $\Delta_{\mathcal{H},\varepsilon}$ is symmetric for some $\varepsilon > 0$. We now turn to proving the McKean-Singer type theorem

Theorem

For every $t > 0$,

$$\begin{aligned} \mathrm{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}}) &= \int_{\mathbb{M}} \mathrm{tr}(p^{+}_{\mathcal{H},\varepsilon}(t,x,x)) - \mathrm{tr}(p^{-}_{\mathcal{H},\varepsilon}(t,x,x)) \, d\mu(x) \\ &= \dim E_0^{+}(\Delta_{\mathcal{H},\varepsilon}) - \dim E_0^{-}(\Delta_{\mathcal{H},\varepsilon}) \\ &= \chi(\mathbb{M}) \end{aligned}$$

Horizontal heat semigroup

First, we justify the existence of the heat semigroup $e^{t\Delta_{\mathcal{H},\varepsilon}}$ by noting that $\Delta_{\mathcal{H},\varepsilon}$ commutes with the Hodge-deRham Laplacian

$$\Delta_\varepsilon = -d\delta_\varepsilon - \delta_\varepsilon d$$

and that for any λ-eigenspace E_λ of Δ_ε it must hold that

$$\Delta_{\mathcal{H},\varepsilon}(E_\lambda) \subset E_\lambda$$

which implies that $\Delta_{\mathcal{H},\varepsilon}$ is essentially self-adjoint and generates a semigroup with heat kernel $p_{\mathcal{H},\varepsilon}$.

Deformation of Laplacians

For $\theta \in [0,1]$ we introduce the operator

$$\Box_{\varepsilon,\theta} = (1-\theta)\Delta_{\mathcal{H},\varepsilon} - \theta D_\varepsilon^2$$

with the Dirac operator $D_\varepsilon := d + \delta_\varepsilon$.

This operator will allow us to carry a few key properties from the Hodge-deRham Laplacian to $\Delta_{\mathcal{H},\varepsilon}$.

Some key propositions

Proposition

Let E_λ^+ (E_λ^-) be the λ-eigenspace of $\Box_{\varepsilon,\theta}$ for even (odd) forms. Then for $\lambda > 0$

$$D_\varepsilon \colon E_\lambda^+ \to E_\lambda^-$$

is an isomorphism.

Proposition

For every $t > 0$, the map

$$\theta \mapsto \mathrm{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}})$$

is continuous.

Proof of McKean-Singer theorem

It follows that

$$\begin{aligned}\mathrm{Str}(e^{t\,\square_{\varepsilon,\theta}}) &= \sum_{\lambda\in\sigma(\square_{\varepsilon,\theta})} \left(\dim E_\lambda^+ - \dim E_\lambda^-\right) e^{t\lambda} \\ &= \dim E_0^+ - \dim E_0^- \in \mathbb{Z}\end{aligned}$$

is constant for both t and θ, and then

$$\mathrm{Str}(e^{t\Delta_{\mathcal{H},\varepsilon}}) = Str(e^{t\Delta_\varepsilon}) = \chi(\mathbb{M})$$

using the usual Hodge theory for the final equality.

Short-time asymptotics

Our last goal is to consider the short time asymptotic,

$$\lim_{t \to 0^+} \operatorname{Str}(p_{\mathcal{H},\varepsilon}(t, x, x))$$

which we will approach using the Brownian-Chen series parametrix method (Boudoin 2008), which is particularly well adapted to hypoelliptic operators.

Brownian-Chen series parametrix

Choosing an orthonormal frame $\{X_i\}$ for $(\mathbb{M}, g)$, for a word $I = (i_1, \dots, i_k)$ and a connection D we define

$$\begin{aligned} D_I &= [D_{X_{i_1}}, [\dots, [D_{X_{i_{k-1}}}, D_{X_{i_k}}] \dots]] \\ X_I &= [X_{i_1}, [\dots, [X_{i_{k-1}}, X_{i_k}] \dots]] \\ \mathcal{F}_I &= D_I - D_{X_I} \end{aligned}$$

Brownian-Chen series parametrix

Consider the operators

$$(P_t^N f)(x) = \mathbb{E}(\Psi(1,x))$$

where $\Psi(\tau, x)$ solves the stochastic differential equation

$$\frac{\partial \Psi}{\partial \tau} = \sum_{I:\ d(I)<N} \Lambda_I(B)_t (D_I \Psi)(\tau, x), \quad \Psi(0,\tau) = f(x)$$

$$\Lambda_I(B)_t := 2^{d(I)/2} \sum_{\sigma \in \mathfrak{G}_k|} \frac{(-1)^{e(\sigma)}}{k^2 \binom{k-1}{e(\sigma)}} \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}(I)}$$

Brownian-Chen series parametrix

Then P_t^N admits a smooth kernel $p^N(t,x,\cdot)$. Moreover, for the operator

$$\mathcal{L} = D_0 + \sum_i D_i^2$$

with heat kernel $p(t,x,\cdot)$ we have the approximation as $t \to 0^+$

$$\begin{aligned} p(t,x,x) &= p^N(t,x,x) + O\left(t^{\frac{N+1-Q}{2}}\right) \\ &= d_t^N(x)\mathbb{E}\left(\exp\left(\sum_{I:\ d(I)\le N} \Lambda_I(B)_t \mathcal{F}_I\right)(x) \middle| \right. \\ &\qquad \left. \sum_{I:\ d(I)\le N} \Lambda_I(B)_t X_I(x) = 0\right) + O\left(t^{\frac{N+1-Q}{2}}\right) \end{aligned}$$

Horizontal Laplacian

Applying this to our situation, we find that if both $n = \operatorname{rank}\mathcal{H}$, $m = \operatorname{rank}\mathcal{V}$ are even, then

$$\lim_{t\to 0^+} \operatorname{Str}(p_{\mathcal{H},\varepsilon}(t,x,x))\ dx = \hat{\omega}^{\varepsilon}_{\mathcal{H}} \wedge \left[\det\left(\frac{\mathcal{T}}{\sinh \mathcal{T}}\right)^{1/2}\right]_m$$

where

$$\mathcal{T}(Y_1, Y_2) := \hat{R}^{\varepsilon}(\pi_{\mathcal{H}} Y_1, Y_2)\pi_{\mathcal{V}}$$

$$\hat{\omega}^{\varepsilon}_{\mathcal{H}} := \frac{(-1)^{n/2} m!}{2^{n/2}\left(\frac{n}{2}+m\right)!}\mathcal{J} \sum_{\sigma,\tau\in\mathfrak{S}_k} e(\sigma)e(\tau)\prod_{i=1}^{n-1} \hat{R}^{\varepsilon,\tau(i+1)}_{\sigma(i)\sigma(i+1)\tau(i)}\ dx_{\mathcal{H}}$$

Horizontal Chern-Gauss-Bonnet Theorem

Collecting the results,

Theorem

If either $n = \operatorname{rank}\mathcal{H}, m = \operatorname{rank}\mathcal{V}$ *are odd, then* $\chi(\mathbb{M}) = 0$*. If they are both even then*

$$\chi(\mathbb{M}) = \int_{\mathbb{M}} \hat{\omega}^{\varepsilon}_{\mathcal{H}} \wedge \left[\det\left(\frac{\mathcal{T}}{\sinh \mathcal{T}}\right)^{1/2}\right]_m .$$

We emphasize that this expression depends only on horizontal curvature quantities, and can be understood as a purely sub-Riemannian result.