The Propagation of Polarization Sets for First-Order Pseudodifferential Systems with Self-Intersecting Characteristics

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Content

1. Definition of polarization sets
2. Propagation of polarization sets for systems of real principal type
3. Propagation of polarization sets for systems of uniaxial type
4. Propagation of polarization sets for systems of generalized uniaxial type (resp. of MHD type)
5. Example: The ideal MHD equations

Introduction

• Hörmander proved that for $u \in \mathcal{D}'(X)$, and $P \in \Psi^m(X)$ of real principal type, we have that $\mathrm{WF}(u) \setminus \mathrm{WF}(Pu) \subseteq \mathrm{Char}\, P$ is invariant under the bicharacteristic flow of P.

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• Dencker studied the propagation of singularities for $P \in \Psi^m(X)$ having characteristics of variable multiplicity. He considered the characteristic set to be union of hypersurfaces S_j, $j = 1, \ldots, r_0$ tangent at $\cap_{j=1}^{r_0} S_j$. Under some assumptions he proved that $\mathrm{WF}(u) \setminus \mathrm{WF}(Pu)$ is invariant under the union of the Hamilton flows on S_j, $j = 1, \ldots r_0$.

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5. Example: The ideal MHD equations

Definition of polarization sets

• Polarization sets refine the notion of the wavefront set for vector-valued distributions.
i.e $u \in \mathcal{D}'(X; \mathbb{C}^N)$ which means $u = (u_j)_{j=1,\dots,N}$ where $u_j \in \mathcal{D}'(X)$
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• $\mathrm{Pol}(u) = \bigcap_{Au \in \mathcal{C}^\infty(X)} \mathcal{N}_A$ where

$$\mathcal{N}_A = \{(x,\xi;w) \in (T^*X \setminus 0) \times \mathbb{C}^N; w \in \ker a(x,\xi)\},$$

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• $\pi(\mathrm{Pol}(u) \setminus 0) = \mathrm{WF}(u)$ where π is the projection on T^*X

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Real principal type

For Scalar Case

$P \in \Psi^m(X)$ is of real principal type if the principal symbol $p(x,\xi)$ is real and the Hamilton field $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ is no where radial i.e $H_p \nparallel R = \xi \frac{d}{d\xi}$.

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Case of system of pseudodifferential operators

An $N \times N$ system P of pseudodifferential operators on X with principal symbol $p(x, \xi)$ is of real principal type at $(y, \eta) \in T^*X \setminus 0$ if $\exists$ an $N \times N$ symbol $\tilde{p}(x, \xi)$ such that

$$\tilde{p}(x, \xi) p(x, \xi) = q(x, \xi) \cdot \mathrm{Id}_N$$

in a neighborhood of (y, η) where $q(x, \xi)$ is a scalar symbol of real principal type and Id_N is the identity in $\mathbb{C}^N$.

Some tools

• Let P be an $N \times N$ system of pseudodifferential operators of order m on a manifold X. The symbol of P is an asymptotic sum of homogeneous terms: $p(x,\xi) + p_{m-1}(x,\xi) + p_{m-2}(x,\xi) + \ldots$, where $p = \sigma(P)$ and p_j is homogeneous of degree j. Assume that P is of real principal type.

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• Let

$$D_P w = H_q w + \frac{1}{2}\{\tilde{p}, p\} w + i\tilde{p} p^s_{m-1} w, \tag{1}$$

with $p^s_{m-1} = p_{m-1} - (2i)^{-1} \sum \partial_{x_j} \partial_{\xi_j} p$, $\{\tilde{p}, p\} = H_{\tilde{p}} p$, and w is a C^∞ function on $T^*X \setminus 0$ with values in $\mathbb{C}^N$.

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with $p^s_{m-1} = p_{m-1} - (2i)^{-1} \sum \partial_{x_j} \partial_{\xi_j} p$, $\{\tilde{p}, p\} = H_{\tilde{p}} p$, and w is a $\mathcal{C}^\infty$ function on $T^*X \setminus 0$ with values in $\mathbb{C}^N$.

• Let $\Sigma = \{(x,\xi) \in T^*X \setminus 0, \det p(x,\xi) = 0\}$ be the characteristic set of P.

Propagation of polarization sets for systems of real principal type

Definition: A Hamilton orbit of a system P of real principal type is a line bundle $L \subseteq \mathcal{N}_P|_\gamma$, where

- γ is an integral curve of the Hamilton field of Σ
- L is spanned by a $\mathcal{C}^\infty$ section w satisfying $D_P w = 0$

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Theorem: Let P be an $N \times N$ system of pseudodifferential operators on a manifold X, and let $u \in \mathcal{D}'(X, \mathbb{C}^n)$. Assume that P is of real principal type at $(y, \eta) \in \Sigma$ and that $(y, \eta) \notin \mathrm{WF}(Pu)$. Then over a neighborhood of $(y, \eta) \in \Sigma$, $\mathrm{Pol}(u)$ is the union of Hamilton orbits of P.

Plan

Assumptions

- $\Sigma = S_1 \cup S_2$, where S_1 and S_2 are non-radial hypersurfaces tangent at $\Sigma_2 = S_1 \cap S_2$ of exactly order $k_0 \geq 1$
- Σ_2 is an involutive manifold of codimension $d_0 \geq 2$
- the dimension of the fiber of $\mathcal{N}_P$ is equal to 2 at Σ_2
- $d^2(\det p) \neq 0$ at Σ_2

Let $\mathcal{N}_P^j = \mathcal{N}_P|_{S_j \setminus \Sigma_2}$, $T_{\Sigma_2}\Sigma = T_{\Sigma_2}S_1 = T_{\Sigma_2}S_2$, and $\partial\Sigma_1 = T_{\Sigma_2}\Sigma / T\Sigma_2$

Limit polarizations

Definition: For $j = 1, 2$, we define the limit polarizations

$$\partial \mathcal{N}_P^j = \{(w, \rho, z) \in \partial \Sigma_1 \times \mathbb{C}^N : \rho \neq 0 \text{ and } z = \lim_{k \to \infty} z_k\},$$

where $z_k \in \ker p(w_k)$ and $w_k \in S_j \setminus \Sigma_2$ satisfy $(w - w_k)/|w - w_k| \to \rho/|\rho|$ when $k \to \infty$.

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- Assume that the fiber of

$$\partial \mathcal{N}_P^1 \cap \partial \mathcal{N}_P^2 = \{0\} \quad \text{over} \quad \partial \Sigma_1 \setminus (\Sigma_2 \times 0).$$

This condition means that no element in $\mathcal{N}_P|_{\Sigma_2}$ can be the limit of polarization vectors on both characteristic surfaces along the same direction.

Propagation of polarization sets for systems of uniaxial type

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Theorem: Let $P \in \Psi^m_{\mathrm{phg}}$ be an $N \times N$ system of uniaxial type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{\mathrm{phg}}$ be a $1 \times N$ system such that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ such that $w_0 \notin \mathrm{WF}(Pu)$, and $w_0 \notin \mathrm{WF}(Au)$. Then, $\mathrm{Pol}(u)$ is a union of $\mathcal{C}^\infty$ line bundles in $\mathcal{N}_A \cap \mathcal{N}_P$ over bicharactersitics of Σ in $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ near w_0, where $\pi_1 : T^*X \setminus 0 \times \mathbb{C}^N \to T^*X$.

Note: At $\Sigma \setminus \Sigma_2$, our system is of real principal type.

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Assumptions

- $\Sigma = \cup_{j=1}^{r_0} S_j$, $r_0 \geq 2$, where S_j are non radial hypersurfaces tangent at $\Sigma_2 = \cap_{j=1}^{r_0} S_j$ of exactly order $k_0 \geq 1$
- Σ_2 is an involutive manifold of codimension $d_0 \geq 2$
- the dimension of the fiber of $\mathcal{N}_P$ is equal to r_0 (resp. $r_0 + 1$) at Σ_2
- $d^{r_0}(\det p) \neq 0$ (resp. $d^{r_0+1}(\det p) \neq 0$)at Σ_2 and $d^i(\det p) = 0$ at Σ_2 if $i < r_0$ (resp. $i < r_0 + 1$)
- the fiber of $\partial\mathcal{N}_P^i \cap \partial\mathcal{N}_P^j = \{0\}$ over $\partial\Sigma_1 \setminus (\Sigma_2 \times 0)$ for $i \neq j$

We say that a system is of generalized uniaxial type if it satisfies the above assumptions (resp. of MHD type)

Propagation of polarization sets for systems of generalized uniaxial type

For systems of generalized uniaxial type we have same result for the propagation of polarization sets as for systems of uniaxial type.

Propagation of polarization sets for systems of MHD type

Theorem: Let $P \in \Psi^m_{\mathrm{phg}}$ be an $N \times N$ system of MHD type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{\mathrm{phg}}$ be a $1 \times N$ system such that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0. Assume that $u \in \mathcal{D}'(X, \mathbb{C}^N)$ such that $w_0 \notin \mathrm{WF}(Pu)$, $w_0 \notin \mathrm{WF}(AD_t u)$, and $[P, D_t]u \in \mathcal{C}^\infty$ at w_0. Then, $\mathrm{Pol}(D_t u)$ is a union of $\mathcal{C}^\infty$ line bundles in $\mathcal{N}_A \cap \mathcal{N}_P$ over bicharactersitics of Σ in $M_A = \pi_1(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ near w_0, where $\pi_1 : T^*X \setminus 0 \times \mathbb{C}^N \to T^*X$.

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The ideal MHD equations

The set of equations describing the ideal MHD are

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla p + H \times \operatorname{curl} H = 0, \\ \partial_t H + u \nabla H + (\operatorname{div} u) H - H \cdot \nabla u - u \operatorname{div} H = 0, \\ \partial_t p + u \cdot \nabla p + \gamma p \operatorname{div} u = 0, \end{cases} \tag{2}$$

where ρ, $p \in \mathbb{R}$ denotes the density and the pressure respectively. $u \in \mathbb{R}^3$ is the fluid velocity, $H \in \mathbb{R}^3$ is the magnetic field, and γ is the adiabatic index.

The linearized ideal MHD equations

Assume a stationary equilibrium the linearized equation of (2) about (ρ, H, p) is:

$$\partial_t \dot{\rho} = -\rho \operatorname{div} \dot{u}, \tag{3a}$$

$$\rho \partial_t \dot{u} = -\nabla \dot{p} + (\nabla \times \dot{H}) \times H, \tag{3b}$$

$$\partial_t \dot{H} = \nabla \times (\dot{u} \times H), \tag{3c}$$

$$\partial_t \dot{p} = -\gamma p \operatorname{div} \dot{u}, \tag{3d}$$

where (ρ, H, p) are the values in the equilibrium state (i.e the solutions of the Ideal MHD equations when $\partial/\partial t = 0$, and as we assumed stationary equilibrium we have $u = 0$).

The characteristic set of the linearized ideal MHD equations

The determinant of the principal symbol of the linearized ideal MHD system is

$$\tau^2(\tau^2 - c_s^2|\xi|^2)(\tau^2 - c_f^2|\xi|^2)(\tau^2 - (\xi \cdot H)^2/\rho) \tag{4}$$

where

$$c_f^2(x,\xi) := \frac{1}{2}\left((c^2+h^2)\xi^2 + \sqrt{(c^2-h^2)^2\xi^4 + 4b^2c^2\xi^2}\right), \tag{5}$$

$$c_s^2(x,\xi) := \frac{1}{2}\left((c^2+h^2)\xi^2 - \sqrt{(c^2-h^2)^2\xi^4 + 4b^2c^2\xi^2}\right), \tag{6}$$

with $c^2 = \gamma p/\rho > 0$, $h^2 = |H|^2/\rho$, $b^2 = |\xi \times H|^2/\rho$. Assume that $0 < |H|^2 \neq \rho c^2$. Hence, we have $c_f^2 > 0$, and $c_f^2 \neq c_s^2$.

Three different cases

First case: If $\tau \neq 0$, and $\tau^2 \neq c_f^2|\xi|^2$, we get $\Sigma = S_1 \cup S_2$, where $S_1 = \{\tau^2 - c_s^2|\xi|^2 = 0\}$, and $S_2 = \{\tau^2 - (\xi \cdot H)^2/\rho = 0\}$.

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Second case: If $\tau^2 \neq c_f^2|\xi|^2$, we get $\Sigma = S_1 \cup S_2 \cup S_3$ where $S_1 = \{\tau^2 - c_s^2|\xi|^2 = 0\}$, $S_2 = \{\tau = 0\}$, and $S_3 = \{\tau^2 - (\xi \cdot H)^2/\rho = 0\}$.

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Third case: If $\tau^2 \neq c_s^2|\xi|^2$, and $\tau \neq 0$, we get $\Sigma = S_1 \cup S_2$, where $S_1 = \{\tau^2 - c_f^2|\xi|^2 = 0\}$, and $S_2 = \{\tau^2 - (\xi \cdot H)^2/\rho\}$.

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Third case: If $\tau^2 \neq c_s^2|\xi|^2$, and $\tau \neq 0$, we get $\Sigma = S_1 \cup S_2$, where $S_1 = \{\tau^2 - c_f^2|\xi|^2 = 0\}$, and $S_2 = \{\tau^2 - (\xi \cdot H)^2/\rho\}$.

In the first and the third case, our system is of uniaxial type. However, in the second case our system is of MHD type.