UNIVERSITY OF BERGEN

Bounded curvature and functional inequalities for hypoelliptic operators

Microlocal and Global Analysis, Interactions with Geometry 2022

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Joint work with Li-Juan Cheng (Hangzhou) and Anton Thalmaier (Luxembourg).

Motivation

Shapes and functional inequalities

Let L be a second order elliptic operator, $P_t = e^{tL/2}$.

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j + \sum_{k=1}^{n} b_k\partial_k, \qquad (a_{ij}) \text{ pos.}, \quad (g_{ij}) = (a_{ij})^{-1}.$$

Let $g = \sum_{i,j=1}^{n} g_{ij}dx_i dx_j$ be the corresponding Riemannian metric.

Shapes and functional inequalities

$$L = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j + \sum_{k=1}^n b_k\partial_k = \Delta_g, \qquad P_t = e^{tL/2}.$$

Let $g = \sum_{i,j=1}^n g_{ij}dx_i dx_j$ be the corresponding Riemannian metric. Then a lower Ricci bound $-K$ of g equivalent to the following inequalities: At any point

$$|\nabla P_t f|^2 \le e^{Kt} P_t|\nabla f|^2$$

$$P_t f^2 - (P_t f)^2 \le \frac{e^{Kt}-1}{K} P_t|\nabla f|^2$$

$$P_t(f^2 \log f^2) - (P_t f^2)\log(P_t f^2) \le \frac{2(e^{Kt}-1)}{K} P_t|\nabla f|^2,$$

and many others

Pinched Ricci

$$L = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j + \sum_{k=1}^n b_k\partial_k, \qquad P_t = e^{tL/2}.$$

$g = \sum_{i,j=1}^n g_{ij}dx_i dx_j$. What if

$$-K \leq \mathrm{Ric} \leq K.$$

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Naber, Cheng, Thalmaier: These are equivalent to functional inequalities involving two points.

Pinched Ricci

$$L = \sum_{i,j=1}^n a_{ij}\partial_i\partial_j + \sum_{k=1}^n b_k\partial_k, \qquad P_t = e^{tL/2}.$$

$g = \sum_{i,j=1}^n g_{ij}dx_i dx_j$. What if

$$-K \leq \mathrm{Ric} \leq K.$$

Naber, Cheng, Thalmaier: These are equivalent to pairs of functional inequalities. Proved by calculations on path space

Main point of todays talk

If L is no longer elliptic, but hypoelliptic, we can still link it to a geometric structure. We can then also find functional inequalities that are equivalent to pinched Ricci curvature. This is in spite of the fact that we are not able to find analogous result in this case for just the lower Ricci bound.

Li-Juan Cheng, G., Anton Thalmaier,.
Functional inequalities on path space of sub-Riemannian manifolds and applications.
Nonlinear Analysis, volume 210, 2021.

Flows of second order operators

Stochastic processes

Consider the solution of the following SDE

$$dX_t^{x,i} = \sum_{j=1}^{d} a_{ij}(X_t)dB_t^j + b^i(X_t)dt, \qquad X_0^x = x.$$

Assume that $(a_{ij}(x))$ is symmetric, positive semi-definite. Write

$$L = \sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j + \sum_k b^k\partial_k.$$

Then

$$P_t f(x) = \mathbb{E}[f(X_t^x)] = e^{tL/2}f(x).$$

Stochastic processes

Suppose we want to study the solution of the following local SDE in a coordinate free way using geometry

$$dX_t^{x,i} = \sum_{j=1}^{d} a_{ij}(X_t)dB_t^j + b^i(X_t)dt, \qquad X_0^x = x.$$

Assume that $(a_{ij}(x))$ is symmetric, positive semi-definite but of rank $k < d$. We can then rewrite the equation as

$$dX_t = \sum_{i=1}^{k} A_k(X_t) \circ dB_t + A_0(X_t)dt, \qquad X_0^x = x$$

for vector some fields $A_0, A_1, \dots, A_k$.

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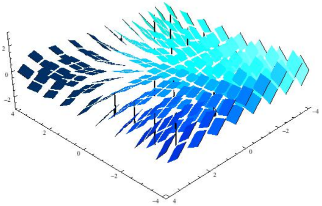
for some vector fields $A_0, A_1, \ldots, A_k$. Define $H = \text{span}\{A_1, \ldots, A_k\}$ and give it a smoothly varying inner product g such that the basis becomes orthonormal.

Sub-Riemannian geometry

$$dX_t = \sum_{i=1}^{k} A_k(X_t) \circ dB_t + A_0(X_t)dt, \qquad X_0^x = x$$

A sub-Riemannian manifold is a triple (M, H, g) where M is a connected manifold, H is a subbundle of TM and g is a metric tensor defined only on H.



Strong Hörmander condition

$H = \mathrm{span}\{A_1, \ldots, A_k\}$. Assume the strong Hörmander condition

$$\mathrm{span}\{A_i, [A_i, A_j], [A_i, [A_j, A_k]], \ldots\} = TM.$$

1. Chow-Rashevskiĭ: Any pair $x, y \in M$ of points can be connected by a curve tangent to H. Taking $d_{cc}(x, y)$ as the infimum of the length of all such curves gives us a well defined distance d_{cc}.
2. Hörmander: The operator

   $$L = \sum_{j=1}^{k} A_j^2 + A_0,$$

   and $\partial_t - L$ are both hypoelliptic.
3. Stroock–Varadhan: The density of X_t is positive everywhere on M.

Rashevskii, P.K
On joining any two points of a completely nonholonomic space by an admissible line.
Uch. Zap. Mosk. Gos. Pedagog. Inst. im. K. Libknekhta. Ser. Fiz.-Mat.,3, No. 2, 83–94 (1938).

Chow, Wei-Liang
Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.
Math. Ann. 117 (1939), 98–105.

Hörmander, Lars,
Hypoelliptic second order differential equations.
Acta Math. 119 (1967), 147–171.

Stroock, Daniel W.; Varadhan, S. R. S..
On the support of diffusion processes with applications to the strong maximum principle.
Proc. of the Sixth Berkeley Symposium on Math. Stat. and Prob., Vol. III: Probability theory, pp. 333–359. Univ. California Press, Berkeley, Calif., 1972.

Example: Heisenberg group

$M = \mathbb{R}^3$, $H = \text{span}\{A_1, A_2\}$, A_1 and A_2 orthonormal basis,

$$A_1 = \partial_x - \tfrac{1}{2} y \partial_z, \qquad A_2 = \partial_y + \tfrac{1}{2} x \partial_z, \qquad A_3 = \partial_z.$$

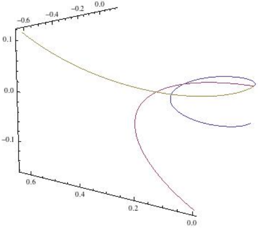
The coordinates $p = (x, y, z)$ as functions

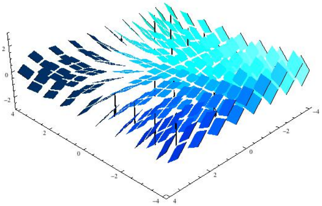
$$x, y \in O\Big(d_{cc}(0, p)\Big), \qquad z \in O\Big(d_{cc}(0, p)^2\Big).$$

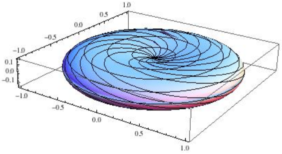
$$A_3 = [A_1, A_2].$$

$\mathbb{R}^3$ with structure

$$H = \text{span}\{A_1, A_2\},$$
$$A_1 = \partial_x - \tfrac{1}{2} y \partial_z,$$
$$A_2 = \partial_y + \tfrac{1}{2} x \partial_z.$$







$$d_{cc}(x,y) = \left\{ \int_0^1 |\dot\gamma(t)|dt \ : \ \begin{array}{c} \gamma \text{ tangent to } H, \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}.$$

As a limit

Let $(M, \bar{g})$ be a Riemannnian manifold with orthogonal decomposition $TM = H \oplus_\perp V$. If we define $g = \bar{g}|_H$ and

$$\bar{g}_\varepsilon = g \oplus \frac{1}{\varepsilon}\bar{g}|_V.$$

then we can consider (M, H, g) as the limit as $\varepsilon \downarrow 0$, that is $d_{cc}(x, y) = \lim_{\varepsilon \downarrow 0} d_\varepsilon(x, y)$. Note: the Ricci curvature of $\bar{g}_\varepsilon$ in general becomes unbounded in the limit.

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1. (M, d_{cc}) independent of the complement V and $\bar{g}|_V$.

A central problem with this geometry

A horizontal curve $[0, T] \to M$, $t \mapsto \omega_t$ is an absolutely continuous curve with $\dot{\omega}_t \in H_{\omega_t}$ for almost every t. In a sub-Riemannian manifold, only horizontal curves have finite length.

A central problem with this geometry

A horizontal curve $[0, T] \to M$, $t \mapsto \omega_t$ is an absolutely continuous curve with $\dot{\omega}_t \in H_{\omega_t}$ for almost every t. In a sub-Riemannian manifold, only horizontal curves have finite length.

Consider a two-parameter family $(s, t) \mapsto \omega_t^s$ such that for each fixed s, the curve $t \mapsto \omega_t^s$ is tangent to H. Then the vector field

$$V(t) = \partial_s \omega_s(t)|_{s=0},$$

does not take values in H (in general). So how do we measure it?

For lower Ricci bounds: Find a good taming metric

For any Riemannian metric $\bar{g}$ is said to tame a sub-Riemannian metric g if $\bar{g}|_H = g$.

For results relating lower Ricci bounds to functional inequalities, usual assumption: "Totally geodesic foliation assumption"

$$(\mathcal{L}_Z \bar{g})(A, A) = 0, \qquad (\mathcal{L}_A \bar{g})(Z, Z) = 0,$$

for A in H and Z in $V = H^{\perp}$.

Several results by

Baudoin, Garofalo, Bonnefont, Wang, Kim, G., Thalmaier, Munive ....

Main inspirations

A. Naber,
Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces,
2015, arXiv:1306.6512v4.

F. Baudoin, Q. Feng, M. Gordina,
Integration by parts and quasi-invariance for the horizontal Wiener measure on foliated compact manifolds.
J. Funct. Anal. 277 (5) (2019), 1362-1422.

Path space: Smooth curves

Analysis of deterministic curves

Sub-Riemannian manifold (M, H, g). Compatible connection ∇.

1. H is preserved under parallel transport,
2. parallel transport restricted to H is an isometry.

Any such connection ∇ has torsion.

$$T(X, Y) = -[X, Y] \mod H, \qquad X, Y \in \Gamma(H).$$

Symmetry lemma

For a torsion-free connection ∇, then for a two-parameter curve ω_t^s, we have

$$D_s \frac{\partial}{\partial t} \omega_t^s = D_t \frac{\partial}{\partial s} \omega_t^s.$$

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For connections with torsion

$$D_s \frac{\partial}{\partial t} \omega_t^s = \hat{D}_t \frac{\partial}{\partial s} \omega_t^s.$$

where $\hat{D}_t$ is the covariant derivative of the adjoint connection

$$\hat{\nabla}_X Y = \nabla_X Y - \mathsf{T}(X, Y).$$

Note: If ∇ is compatible with a sub-Riemannian structure (H, g), $\hat{\nabla}$ will not be.

Development with respect to an arbitrary connection

Let $t \mapsto \omega_t$ be a curve in M starting at x and define

$$//_t : T_x M \to T_{\omega_t} M$$

as parallel transport along ω_t. A curve u_t in $T_x M$ solving

$$\dot{u}_t = //_t^{-1} \dot{\omega}_t, \qquad u_0 = 0.$$

is called the anti-development and ω_t is called the development. Gives one-to-one correspondence

$$C_0([0,\infty), T_x M, 0) \stackrel{\mathrm{Dev}}{\to} C^\infty([0,\infty), M, x).$$
$$\mathrm{Dev}(u)_t = \omega_t$$

What is the derivative of this map?

Derivative of the development map

Lemma

If u_t, k_t are curves in T_xM, and

$$Y_t = \frac{d}{ds}\,\mathrm{Dev}(u+sk)_t\big|_{s=0},$$

then Y_t is the vector field along $\omega_t = \mathrm{Dev}(u_t)$ given by $Y_t = //_t y_t$,

$$k_t = y_t + \int_0^t \mathsf{T}_{//_s}(y_s, du_s) - \int_0^t \int_0^s \mathsf{R}_{//_s}(du_r, y_r)du_s$$

Develop curves in sub-Riemannian manifolds?

We have to pick a connection. We will choose one relative to a choice of complement.

A complement V to a sub-Riemannian manifold is called metric-preserving if

$$(\mathcal{L}_Z g)(A, A) = 0, \qquad A \in \Gamma(H), Z \in \Gamma(V).$$

Develop curves in sub-Riemannian manifolds?

Lemma

If we have a complement $TM = H \oplus V$ that is metric preserving, then there is an (almost) unique connection satisfying the following,

(a) *Both H and V are parallel with respect to ∇,*

(b) *∇ is compatible with (H, g),*

(c) $\mathsf{T}(H, H) \subseteq V$, $\mathsf{T}(H, V) = 0$.

($\nabla_{\mathrm{pr}_V X} \mathrm{pr}_V Y$ can be arbitrary, but we will never use this part.)

Derivative of the development map

Using such a connection, we have

Lemma

If u_t, k_t are curves in H_x, and

$$Y_t = \frac{d}{ds}\,\mathrm{Dev}(u+sk)_t|_{s=0},$$

then Y_t is the vector field along $\omega_t = \mathrm{Dev}(u)_t$ given by $Y_t = //_t y_t$, where

$$y_t = h_t + \int_0^t dA_s h_s, \qquad h_t \in H_x \qquad A_t w = \int_0^t \mathsf{T}_{//_s}(du_s, w),$$

with

$$k_t = h_t - \int_0^t \int_0^s \mathsf{R}_{//_s}(du_r, h_r)du_s$$

Path space of a sub-Riemannian manifold

Recall: Malliavin calculus

Malliavin calculus: Taking derivatives of a random variable: Let $\xi_k(u) = u + k$. Then

$$\frac{d}{d\varepsilon}\mathbb{E}[F \circ \xi_{\varepsilon k}(X_t)]|_{\varepsilon=0} \overset{Girsanov}{=} \mathbb{E}\left[F(X_t)\int_0^T \langle \dot{k}_t, dB_t\rangle\right],$$

where k is in the Cameron-Martin space: absolutely continuous $k_0 = 0$, $\int_0^T |\dot{k}_t|^2 dt < \infty$.

If F is a cylinder function $F(\omega) = f(\omega_{t_1}, \ldots, \omega_{t_n})$, then

$$\frac{d}{d\varepsilon}\mathbb{E}[F \circ \xi_{\varepsilon k}(X_t)]|_{\varepsilon=0} = \sum_{i=1}^n \langle d_i f, k_{t_i}\rangle = \int_0^t \langle D_t f, \dot{k}_t\rangle dt,$$

where

$$D_t F = \sum_{i=1}^n 1_{t \le t_i} \sharp d_i f.$$

Recall: Malliavin calculus

On Riemannian manifolds: Use parallel transport with respect to the Levi-Civita connection.

If $\sharp : T^*M \to TM$ is the identification using the metric, then the gradient is defined as

$$D_t F = \sum_{i=1}^{n} 1_{t \le t_i} //_t^{-1} \sharp d_i f.$$

but also the damped gradient

$$\tilde{D}_t F = \sum_{i=1}^{n} 1_{t \le t_i} \sharp //_t^{-1} Q_{t,t_i}^* //_t^{-1} d_i f.$$

$$\frac{d}{dt} Q_{s,t} = -\frac{1}{2} \operatorname{Ric}_{//_{s,t}} Q_{s,t}, \qquad Q_{s,s} = \mathrm{id}_{T_{X_s} M} .$$

Recall: Malliavin calculus

$$\tilde{D}_t F = \sum_{i=1}^{n} \sharp //_t^{-1} Q_{t,t_i} //_t^{-1} d_i f, \qquad \frac{d}{dt} Q_{s,t} = -\frac{1}{2} \operatorname{Ric}_{//_{s,t}} Q_{s,t}, .$$

Then

$$\frac{d}{d\varepsilon} \mathbb{E}[(F \circ \operatorname{Dev} \circ \xi_{\varepsilon k} \circ \operatorname{Dev}^{-1})(X_t)]|_{\varepsilon=0} = \mathbb{E}\left[F(X_t) \int_0^T \langle \dot{k}_t, dB_t \rangle \right]$$

$$= \int_0^T \langle \tilde{D}_t F, \dot{k}_t \rangle dt =: \tilde{D}_k F.$$

Now to the sub-Riemannian case

… but first some notation: For a sub-Riemannian manifold (M, H, g), there still exist a map $\sharp : T^*M \to H \subseteq TM$,

$$\langle \alpha, v \rangle = \langle \sharp\alpha, v \rangle_g, \qquad \alpha \in T_xM, v \in H_x.$$

which is not invertible. We also have a cometric

$$\langle \alpha, \beta \rangle_{g^*} = \langle \sharp\alpha, \sharp\beta \rangle_g,$$

which is degenerate.

Sub-Riemannian diffusions

Let B_t^x be the standard Brownian on H_x starting at $0 \in H_x$. Let V be a metric-preserving complement with corresponding connection ∇. Define X_t as the solution of

$$dX_t^x = //_t \circ dB_t^x.$$

Infinitesimal generator

$$L = \frac{1}{2} \operatorname{tr}_H \nabla^2_{\times,\times}.$$

Define the Cameron-Martin space as $\mathbb{H}^T(H_x)$ of absolutely continuous functions $h : [0, T] \to H_x$ such that $h_0 = 0$, $\int_0^T |\dot{h}_t|_g^2 \, dt < \infty$ and with inner product

$$\langle h, k \rangle_{\mathbb{H}} = \int_0^T \langle \dot{h}_t, \dot{k}_t \rangle_g \, dt.$$

Recall: Smooth case

Using such a connection, we have

Lemma

If u_t, k_t are curves in H_x, and

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then Y_t is the vector field along $\omega_t = \mathrm{Dev}(u_t)$ given by $Y_t = //_t y_t$, where

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$$k_t = h_t - \int_0^t \int_0^s \mathbf{R}_{//_s}(du_r, h_r)\,du_s$$

We define a vector field D_h on paths by

$$D_h|_{\omega_t} = //_t\left(h_t + \int_0^t dA_s h_s\right).$$

Gradient on path space

Define a process $A_t : T_x M \to T_x M$, by

$$A_t = \int_0^t \mathrm{T}_{//_s}(\circ dB_s^x, \cdot\,) = \int_0^t \mathrm{T}_{//_s}(dB_s^x, \cdot\,) - \frac{1}{2}\int_0^t (\delta_H \mathrm{T})_{//_s}\, ds.$$

Gradient on path space

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For any $h \in W_0^\infty(H_x)$, define a vector field on $F : \omega \mapsto f(\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n})$ by

$$D_h F = \sum_{i=1}^n \left\langle //_{t_i}^{-1} d_i f|_{\omega_{t_1},\ldots,\omega_{t_n}}, h_{t_i} + \int_0^{t_i} dA_s h_s \right\rangle.$$

We define

$$D_t F = \sum_{i=1}^n 1_{t \le t_i} \sharp(\mathrm{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i f|_{\omega_{t_1},\ldots,\omega_{t_n}}.$$

so that $\int_0^T \langle D_t F, \dot{h}_t \rangle_g dt = D_h F$.

The damped gradient

We define

$$\mathrm{Ric}(v) = -\operatorname{tr}_H \mathrm{R}(\times, v)\times .$$

Use the adjoint connection: Define $\hat{Q}_{s,t} : T_{X_s}M \to T_{X_t}M$ by

$$\hat{Q}_{s,s} = \mathrm{id}, \qquad \frac{d}{dt}\hat{Q}_{s,t} = -\frac{1}{2}\,\mathrm{Ric}_{\hat{//}_{s,t}}\,\hat{Q}_{s,t}.$$

Define

$$\tilde{D}_t F(\omega) = \sum_{i=1}^{n} 1_{t \le t_i} \sharp \hat{//}_t^{-1} \hat{Q}_{t,t_i}^{*} \hat{//}_{t,t_i}^{-1} d_i f|_{\omega_{t_1},\dots,\omega_{t_n}} .$$

and $\tilde{D}_k F = \int_0^T \langle \tilde{D}_t F, k_t \rangle_{\mathbb{H}} dt$. Remember that $\hat{\nabla}$ does not preserve the sub-Riemannian structure.

Give us the identities we want want

$$\text{(Derivative formula)} \qquad D_t \mathbb{E}_x[F|\mathscr{F}_t] = \mathsf{E}_x[\tilde{D}_t F|\mathscr{F}_t],$$

$$\text{(Clark-Ocône)} \qquad F - \mathbb{E}_x[F] = \int_0^T \langle \mathsf{E}_x[\tilde{D}_t F|\mathscr{F}_t], dB_t\rangle,$$

$$\text{(Integration by parts)} \qquad \mathbb{E}_x[\langle \tilde{D}F, k\rangle_{\mathbb{H}}] = \mathbb{E}_x[F\langle k, B^x\rangle_{\mathbb{H}}],$$

.. and a quasi-invariance result

$X_t = \mathrm{Dev}(B_t), \qquad X_t^s = \mathrm{Dev}(B_t - sk_t)$

$$\mathbb{E}_x\left[\langle \tilde{D}F, k\rangle_{\mathbb{H}}\right] = \lim_{s\to 0} \frac{\mathbb{E}[F(X_t^s)] - \mathbb{E}[F(X_t)]}{s}.$$

Ideas of the proof

- We use the relation

$$\tilde{D}_k F = D_h F, \qquad h_t = Q_t \int_0^t Q_s^{-1}\, dk_s, \qquad Q_t = Q_{0,t}.$$

$$//_t^{-1} \hat{//}_t \hat{Q}_t \int_0^t \hat{Q}_s^{-1} \hat{//}_s^{-1} //_s dk_s = h_t + \int_0^t dA_s h_s.$$

- For $P_t f = \mathbb{E}[f(X_t)]$.

$$\tilde{N}_s = \hat{Q}_s^* \hat{//}_s^{-1} dP_{t-s} f|_{X_s^x}.$$

is a martingale in $T_x^* M$. We add usual induction argument for a general cylinder function.

Bounded curvature and functional inequalities

On bounded curvature

Idea: Lower Ricci correspond to inequalities for the heat semi-group, bounded Ricci to inequalities on path space.

Let Ric be defined relative to ∇ using only a trace over the horizontal bundle. Define $\mathrm{Ric}_H = \mathrm{Ric}|_H$. Then for the sub-Riemannian case, we have functional inequalities equivalent to bounded Ricci curvature.

Equivalence to bounded curvature

Theorem

$$-K \leq \mathrm{Ric}_H \leq K.$$

The following are equivalent. For cylinder functions

$$(i) \qquad |D_0 \mathbb{E}_x[F]|_g \leq \mathbb{E}_x \left[|D_0 F|_g + \frac{K}{2} \int_0^T e^{Ks/2} |D_s F|_g ds \right],$$

$$(ii) \qquad |D_0 \mathbb{E}_x[F]|_g^2 \leq e^{KT/2} \mathbb{E}_x \left[|D_0 F|_g^2 + \frac{K}{2} \int_0^T e^{Ks/2} |D_s F|_g^2 ds \right].$$

Equivalence to bounded curvature

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The following are equivalent. For cylinder functions

(iii) $$\mathbb{E}_x\left[\mathbb{E}_x[F^2|\mathscr{F}_t]\log\mathbb{E}_x[F^2|\mathscr{F}_t]\right]-\mathbb{E}_x[F^2]\log\mathbb{E}_x[F^2] \leq 2\int_0^t e^{K(T-r)}\left(\mathbb{E}_x|D_rF|_g^2+\frac{K}{2}\int_r^T e^{K(s-r)/2}\mathbb{E}_x|D_sF|_g^2 ds\right)dr$$

(iv) $$\mathbb{E}_x\left[\mathbb{E}_x[F|\mathscr{F}_t]^2\right]-\mathbb{E}_x[F]^2 \leq \int_0^t e^{k(T-r)/2}\left(\mathbb{E}_x|D_rF|_g^2+\frac{K}{2}\int_r^T e^{K(s-r)/2}\mathbb{E}_x|D_sF|_g^2 ds\right)dr$$

Equivalence to bounded curvature

Theorem

$$-K \le \mathrm{Ric}_H \le K.$$

The following are equivalent. For $f \in C_0(M)$

(v) $$|dP_t f(x)|^2_{g^*} - e^{Kt/2}\mathbb{E}_x\left[\left|(\mathrm{id} + A_t)^* //_t^{-1} //_t^{-1} df(X_t)\right|^2_{g^*}\right] \le \mathbb{E}_x\left[\frac{K}{2}\int_0^t e^{K(t+s)/2}|(\mathrm{id} + A_t - A_s)^* //_t^{-1} df(X_t)|^2_{g^*} ds\right]$$

and

$$|2df - P_t f|^2_{g^*}(x) - e^{Kt/2}\mathbb{E}_x\left[|2df(x) - (\mathrm{id} + A_t)^* //_t^{-1} df(X_t)|^2_{g^*}\right] \le \mathbb{E}_x\left[\frac{K}{2}\int_0^t e^{\frac{K}{2}(t+s)}|(\mathrm{id} + A_t - A_s)^* //_t^{-1} df(X_t)|^2_{g^*} ds\right]$$

Idea of proof

- Similar steps to Naber's paper: Using the previously mentioned properties of the gradients along with estimates of the Ricci curvature.
- We need several specific estimates related to A_t to make ”extra terms” disappear.

Applications

Our definition of the gradient also give us a symmetric form

$$\mathscr{E}_x(F, G) = \mathbb{E}_x \int_0^T \langle D_t F, D_t G \rangle_g \, dt.$$

on a subdomain of $L^2(W^{x,T}, \mathbb{P}^{x,T})$. Defines a corresponding Ornstein-Uhlenbeck operator $\mathscr{L}$.

Applications

Corollary

1. *If* $\mathbb{E}_x[F] = 0$*, then*

$$\mathbb{E}_x[F^2] \leq \frac{1}{2}(e^{KT} + 1)\mathscr{E}_x(F, F).$$

2. *If* $\mathbb{E}_x[F^2] = 1$*, then*

$$\mathbb{E}_x(F^2 \log F^2) \leq (e^{KT} + 1)\mathscr{E}_x(F, F).$$

3. *We have a spectral gap estimate*

$$\mathrm{gap}(\mathscr{L})^{-1} \leq \frac{1}{2}(e^{KT} + 1).$$

Geometric interpretation

In our discussions above, our result depend on (M, H, g) but also a choice of complement V. Unreasonable?

1. Observe that the perturbations $\mathrm{Dev} \circ \xi_h \circ \mathrm{Dev}^{-1}$ depend on the choice of complement, hence also the gradients in our results.
2. Ric_H curvature of the development,
3. If V is integrable, then we can consider it locally as the vertical bundle of $\pi : M \to B$, and g is the pullback of some Riemannian metric on B. Bounds of the horizontal part of the Ricci curvature of ∇ the same as the bounds of the Ricci curvature on B.

Thank you very much!