Boundary behavior of Dirichlet solutions to strongly elliptic fractional-order operators

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1. Strongly elliptic operators.

Operators P of fractional (i.e., not necessarily integer) order are of great interest in probability theory and finance, as well as in in mathematical physics and differential geometry. A highly studied case is the fractional Laplacian $(-\Delta)^a$, $0 < a < 1$.

But also other operators are of interest, for example the fractional Laplacian with drift

$$(-\Delta)^a + \boldsymbol{b} \cdot \nabla, \quad \boldsymbol{b} \in \mathbb{R}^n,$$

especially in the case $a = \frac{1}{2}$, where the two terms are of order 1 but do not mingle well.

For $(-\Delta)^a$ there is a well-established theory on how it acts on functions u on a given domain Ω with u extended by zero in $\mathbb{R}^n \setminus \Omega$. It shows that the solutions of the homogeneous Dirichlet problem

$$Pu = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \qquad (1)$$

for nice f have a structure at the boundary $\partial\Omega$ with a factor $d(x)^a$, $d(x) = \operatorname{dist}(x, \partial\Omega)$. The technical fact that leads to this is that the symbol $|\xi|^{2a}$ is **even** in ξ, taking the same value at ξ and $-\xi$.

In contrast, for $P = (-\Delta)^{\frac{1}{2}} + \boldsymbol{b} \cdot \nabla$, the boundary behavior of solutions to (1) carries a factor $d^{\mu(z)}$ near the boundary, where $\mu(z)$ is defined at each boundary point z depending on the normal $\nu(z)$ to the boundary.

This was shown by Fernandez-Real and Ros-Oton '18. A recent paper by Dipierro, Ros-Oton, Serra and Valdinoci (arXiv '20) treats operators L that are sums of two terms L_e and L_o of order $2a$ having real homogeneous kernels K_e resp. K_o, such that K_e is **even** and K_o is **odd**. Then L_e has **even** real symbol $\mathcal{A}(\xi)$ and L_o has **odd** purely imaginary symbol $i\mathcal{B}(\xi)$. The symbols are x-independent, and $\mathcal{A}(\xi) > 0$ for $\xi \neq 0$.

We shall go a step further, and study operators with completely general, strongly elliptic symbols; homogeneous and x-independent. We make an analysis of how they act on $\mathbb{R}^n_+$, the model case.

Recall the definition of a *pseudodifferential operator* (ψdo) $P = \mathrm{Op}(p(x,\xi))$; with *symbol* $p(x,\xi)$. Denote by $\mathcal{F}$ the Fourier transform:

$$\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx,$$

then

$$Pu(x) = \mathcal{F}^{-1}_{\xi \to x}\big(p(x,\xi)\hat{u}(\xi)\big).$$

When independent of x, $\mathrm{Op}(p(\xi))\,\mathrm{Op}(q(\xi)) = \mathrm{Op}(p(\xi)q(\xi))$ simply.

Dipierro, Ros-Oton, Serra and Valdinoci show the following:

Theorem 1. *Let Ω be a bounded $C^{1,\alpha}$-domain. There is a real Hölder continuous function $\mu(z)$ on the boundary, defined from the value of the symbol on the interior normal $\nu(z)$, such that if u is a bounded solution of (1) with $f \in L_\infty(\Omega)$, then $u \in d^{\mu(z)} C^\varepsilon(\overline{\Omega})$ for some $\varepsilon > 0$.*
There is an integration by parts formula for such solutions u together with analogous solutions u' of the problem for the adjoint P^*, for each direction v:

$$\int_\Omega (\partial_v u\, P^* u' + Pu\, \partial_v u')\, dx = \int_{\partial\Omega} s_v(x) \gamma_0\big(\tfrac{u}{d^{\mu(z)}}\big) \gamma_0\big(\tfrac{u'}{d^{\mu'(z)}}\big)\, d\sigma.$$

The method of proof is completely real. A crucial ingredient in the proof is the integration by parts formula over $\mathbb{R}^n_+$ (with a fixed μ), which is worked out using painstaking calculations involving special functions.

The authors invited me to comment on a preliminary version of the manuscript, and we have had interesting discussions. This made me try to attack the problem from a complex point of view, and I shall now show an alternative way to the appearance of the factor d^μ, and the integration by parts formula over $\mathbb{R}^n_+$.

We consider an operator $P = \mathrm{Op}(p(\xi))$ whose symbol satisfies

- $p(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$ is homogeneous of degree $2a > 0$, $0 < a < 1$,
- $\operatorname{Re} p(\xi) > 0$ for $\xi \neq 0$ (strong ellipticity).

Write $x \in \mathbb{R}^n$ as $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and let $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$. For a function u on $\mathbb{R}^n$, $r^+ u$ denotes $u|_{\mathbb{R}^n_+}$, and for a function v on $\mathbb{R}^n_+$, $e^+ v$ denotes the extension by zero on $\mathbb{R}^n_-$.

P is applied to functions on $\mathbb{R}^n_+$ by extending them by zero to $\mathbb{R}^n$, applying P, and restricting back to $\mathbb{R}^n_+$. The *homogeneous Dirichlet problem* for P on $\mathbb{R}^n_+$ is usually written:

$$Pu = f \text{ on } \mathbb{R}^n_+, \quad u = 0 \text{ on } \mathbb{R}^n_-, \qquad (1)$$

where the restriction-and-extension operators are understood. To discuss solution properties, introduce Sobolev spaces for $s \in \mathbb{R}$:

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_2(\mathbb{R}^n)\}, \\ \overline{H}^s(\mathbb{R}^n_+) &= r^+ H^s(\mathbb{R}^n), \qquad \text{the } \textit{restricted} \text{ space}, \\ \dot{H}^s(\overline{\mathbb{R}}^n_+) &= \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\mathbb{R}}^n_+\}, \text{ the } \textit{supported} \text{ space}; \end{aligned}$$

here $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. $\dot{H}$ is often called $\tilde{H}$ or H_0; the dot and overline notation stems from Hörmander's books.

There is an interesting treatment of the halfspace case in the book of Eskin (English translation '81) based on Vishik and Eskin's work in the '60ies. We borrow a trick from there: To handle the singularity at $\xi = 0$, introduce when $a(\xi)$ is homogeneous of order m:

$$\widehat{a}(\xi) = a(\langle \xi' \rangle \tfrac{\xi'}{|\xi'|}, \xi_n); \text{ then } a'(\xi) = a(\xi) - \widehat{a}(\xi) \text{ is } O(|\xi|^{m-1}) \text{ for } |\xi| \to \infty,$$

and $\widehat{a}$ is continous on $\mathbb{R}^n$. Do calculations on $\widehat{a}$ and return to a in the conclusions.
By the strong ellipticity of our P, we have that $\operatorname{Re} \widehat{p}(\xi) \geq c > 0$ on $\mathbb{R}^n$. Then $\widehat{P}$ is associated with a positive sesquilinear form on $C_0^\infty(\mathbb{R}^n_+)$,

$$s(u,v) = \int_{\mathbb{R}^n_+} \widehat{P} u \, \bar{v} \, dx$$

extended by continuity to $\dot{H}^a(\overline{\mathbb{R}}^n_+)$, which induces by the Lax-Milgram lemma an operator $\widehat{P}_D$ with domain

$$D(\widehat{P}_D) = \{ u \in \dot{H}^a(\overline{\mathbb{R}}^n_+) \mid r^+ \widehat{P} u \in L_2(\mathbb{R}^n_+) \};$$

it is bijective from $D(\widehat{P}_D)$ onto $L_2(\mathbb{R}^n_+)$. Thus (1) is uniquely solvable for this operator when $f \in L_2(\mathbb{R}^n_+)$. What more can be said about u, when $f \in L_2(\mathbb{R}^n_+)$ or in better spaces $\overline{H}^s(\mathbb{R}^n_+)$?

2. Transmission spaces, a simple example.

To motivate transmission spaces, we recall a very simple case:

Example. Let $P = (-\Delta)^a$, $0 < a < 1$, with symbol $p(\xi) = |\xi|^{2a} = (|\xi'|^2 + \xi_n^2)^a$. Here $\widehat{p}(\xi) = (\langle \xi' \rangle^2 + \xi_n^2)^a = (1 + |\xi|^2)^a$, and $\widehat{P} = (1 - \Delta)^a$. For the operator $\widehat{P}$ there is an easy theory. Since

$$(\langle \xi' \rangle^2 + \xi_n^2)^a = (\langle \xi' \rangle - i\xi_n)^a (\langle \xi' \rangle + i\xi_n)^a,$$

we have by defining

$$\Xi_\pm^t = \mathrm{Op}((\langle \xi' \rangle \pm i\xi_n)^t), \quad t \in \mathbb{C},$$

that $\widehat{P} = (1 - \Delta)^a$ has the factorization

$$(1 - \Delta)^a = \Xi_-^a \, \Xi_+^a, \text{ with inverse } (1 - \Delta)^{-a} = \Xi_+^{-a} \, \Xi_-^{-a}.$$

Relative to $\mathbb{R}_+^n$, the operators $\Xi_\pm^t$ map as follows for all $s \in \mathbb{R}$:

$$\Xi_+^t \colon \dot{H}^s(\overline{\mathbb{R}}_+^n) \overset{\sim}{\to} \dot{H}^{s - \operatorname{Re} t}(\overline{\mathbb{R}}_+^n), \quad r^+ \Xi_-^t e^+ \colon \overline{H}^s(\mathbb{R}_+^n) \overset{\sim}{\to} \overline{H}^{s - \operatorname{Re} t}(\mathbb{R}_+^n).$$

Then over $\mathbb{R}_+^n$, the variational solution operator to the Dirichlet problem for $(1 - \Delta)^a$ maps with bijective factors:

$$\overline{H}^{-a}(\mathbb{R}_+^n) \overset{r^+ \Xi_-^{-a} e^+}{\longrightarrow} L_2(\mathbb{R}_+^n) \overset{\Xi_+^{-a}}{\longrightarrow} \dot{H}^a(\overline{\mathbb{R}}_+^n).$$

When we lift this to higher-order Sobolev spaces, so-called transmission spaces comes in:

$$\overline{H}^s(\mathbb{R}^n_+) \overset{r^+\Xi_-^{-a}e^+}{\longrightarrow} \overline{H}^{s+a}(\mathbb{R}^n_+) \overset{\Xi_+^{-a}}{\longrightarrow} \Xi_+^{-a}e^+\overline{H}^{s+a}(\mathbb{R}^n_+) \equiv H^{a(s+2a)}(\overline{\mathbb{R}}^n_+).$$

The range space here is the a-transmission space, originally introduced in a lecture note by Hörmander '66.
Define generally for $\operatorname{Re}\mu > -1$, $t - \operatorname{Re}\mu > -\frac{1}{2}$, the μ-transmission space

$$H^{\mu(t)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-\mu}e^+\overline{H}^{t-\operatorname{Re}\mu}(\mathbb{R}^n_+).$$

It equals $\dot{H}^t(\overline{\mathbb{R}}^n_+)$ when $-\frac{1}{2} < t - \operatorname{Re}\mu < \frac{1}{2}$, and satisfies

$$H^{\mu(t)}(\overline{\mathbb{R}}^n_+) \subset \dot{H}^t(\overline{\mathbb{R}}^n_+) + x_n^\mu e^+\overline{H}^{t-\operatorname{Re}\mu}(\mathbb{R}^n_+) \text{ when } t - \operatorname{Re}\mu > \tfrac{1}{2}.$$

It contains $e^+x_n^\mu\mathcal{S}(\overline{\mathbb{R}}^n_+)$ as a dense subset. We have obtained:

Proposition 1. *The solutions u of the homogeneous Dirichlet problem* (1) *for $(1-\Delta)^a$ on $\mathbb{R}^n_+$ satisfy for $s \geq 0$:*

$$f \in \overline{H}^s(\mathbb{R}^n_+) \iff u \in H^{a(s+2a)}(\overline{\mathbb{R}}^n_+). \quad (2)$$

Note that u has a factor x_n^a when $s + a > \frac{1}{2}$. Now go to more general operators P.

3. General P.

For a general symbol $p(\xi)$, homogeneous of degree $2a$, we look for a factorization

$$P = \Xi_{0,-}^{\mu'} Q \, \Xi_{0,+}^{\mu}, \quad p(\xi) = (|\xi'| - i\xi_n)^{\mu'} q(\xi) \, (|\xi'| + i\xi_n)^{\mu},$$
$$\widehat{P} = \Xi_{-}^{\mu'} \widehat{Q} \, \Xi_{+}^{\mu}, \quad \widehat{p}(\xi) = (\langle\xi'\rangle - i\xi_n)^{\mu'} \widehat{q}(\xi) \, (\langle\xi'\rangle + i\xi_n)^{\mu},$$

where $\mu' = 2a - \mu$ so that Q and $\widehat{Q}$ are of order 0. Here μ should be chosen so that

$$q(0,-1) = q(0,1), \qquad (3)$$

i.e., the symbol is **even** in the ξ_n-direction; then Q and $\widehat{Q}$ have nice mapping properties with respect to the ordinary Sobolev spaces over $\mathbb{R}^n_\pm$. For (3), $\mu \in \mathbb{C}$ can be chosen uniquely mod $\mathbb{Z}$. There is an optimal choice $\mu = a + \delta$ with $|\operatorname{Re}\delta| < \frac{1}{2}$, which assures that $-\pi < \arg q(\xi) < \pi$; we use that here.

Those who know the Boutet de Monvel calculus, will recognize (3) as the first step in the 0-*transmission condition*. The full 0-transmission condition is:

$$\partial^\alpha q(0,-1) = (-1)^{|\alpha|} \partial^\alpha q(0,1), \quad (4)$$

We call (3) the *principal* 0-*transmission condition*.

(4) will not hold in general, but a few steps of good mapping properties can be shown on the basis of (3). By use of the Wiener-Hopf decomposition principle (recalled in Eskin '81), one can show:

Proposition 2. *When q is homogeneous of degree 0 and nonvanishing, with $-\pi < \arg q(\xi) < \pi$, and (3) holds, then there is a factorization,*

$$q(\xi', \xi_n) = q^-(\xi', \xi_n) q^+(\xi', \xi_n),$$

where $q^\pm$ have holomorphic extensions in ξ_n in $\mathbb{C}_-$ resp. $\mathbb{C}_+$.

Symbol estimates for $q^\pm$ can be derived using a lemma in Eskin '81. Let $\widehat{Q}^\pm = \mathrm{Op}(\widehat{q}^\pm(\xi))$; they have the properties

$$\widehat{Q}^+ \colon \dot{H}^s(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} \dot{H}^s(\overline{\mathbb{R}}^n_+), \quad r^+ \widehat{Q}^- e^+ \colon \overline{H}^s(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{H}^s(\mathbb{R}^n_+), \text{ for } s \in \mathbb{R}.$$

Then $r^+ \widehat{Q} e^+$ acts over $\mathbb{R}^n_+$ like $(r^+ \widehat{Q}^- e^+)(r^+ \widehat{Q}^+ e^+)$. It maps $\overline{H}^s(\mathbb{R}^n_+)$ to itself for $|s| < \frac{1}{2}$, since $\overline{H}^s(\mathbb{R}^n_+) = \dot{H}^s(\overline{\mathbb{R}}^n_+)$ for such s.

The crucial observation is now that the symbol of $\widehat{Q}^+$ is such that the map can be extended to the values $\frac{1}{2} \le s < \frac{3}{2}$, implying

Theorem 3. *When q is as in Proposition 2, then*

$$r^+ \widehat{Q} e^+ \colon \overline{H}^s(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{H}^s(\mathbb{R}^n_+) \text{ for } -\tfrac{1}{2} < s < \tfrac{3}{2}.$$

Going back to $\widehat{P}$, we see that $r^+\widehat{P}$ over $\overline{\mathbb{R}}^n_+$ is composed of three factors:

$$r^+\widehat{P} = (r^+\Xi_-^{\mu'} e^+)(r^+\widehat{Q}e^+)\Xi_+^{\mu}.$$

Here we can invert all three, in the range of spaces where the middle factor is invertible. This gives a solution operator for the Dirichlet problem for $\widehat{P}$:

Theorem 4. *The homogeneous Dirichlet problem for $\widehat{P}$ on $\mathbb{R}^n_+$,*

$$\widehat{P}u = f \text{ on } \mathbb{R}^n_+, \quad u = 0 \text{ on } \mathbb{R}^n_-, \qquad (5)$$

has the solution operator

$$\widehat{R} = \Xi_+^{-\mu}(r^+\widehat{Q}e^+)^{-1}(r^+\Xi_-^{-\mu'} e^+),$$

mapping (with bijective factors) for $-\frac{1}{2} < s + \operatorname{Re}\mu' < \frac{3}{2}$:

$$\overline{H}^s(\mathbb{R}^n_+) \overset{r^+\Xi_-^{-\mu'}e^+}{\longrightarrow} \overline{H}^{s+\operatorname{Re}\mu'}(\mathbb{R}^n_+) \overset{(r^+\widehat{Q}e^+)^{-1}}{\longrightarrow} \overline{H}^{s+\operatorname{Re}\mu'}(\mathbb{R}^n_+) \overset{\Xi_+^{-\mu}}{\longrightarrow} H^{\mu(s+2a)}(\overline{\mathbb{R}}^n_+).$$

Note that μ-transmission spaces show up as range spaces for $\widehat{R}$, i.e. domain spaces for the Dirichlet problem.
Writing μ as $a+\delta$ with $|\operatorname{Re}\delta| < \frac{1}{2}$, we can write the condition on s as $-\frac{1}{2} < s + a - \operatorname{Re}\delta < \frac{3}{2}$. The value $s=0$ is included, allowing $f \in L_2$.

For the original operator P, the fact that $P - \widehat{P}$ is of order $2a-1$ allows to show the regularity theorem:

Theorem 5. *For* $-\frac{1}{2} < s + a - \operatorname{Re}\delta < \frac{3}{2}$, r^+P *maps* $H^{\mu(s+2a)}(\overline{\mathbb{R}}^n_+)$ *to* $\overline{H}^s(\mathbb{R}^n_+)$, *and the solutions of the Dirichlet problem* (1) *satisfy:*

$$r^+Pu \in \overline{H}^s(\mathbb{R}^n_+) \implies u \in H^{\mu(s+2a)}(\overline{\mathbb{R}}^n_+).$$

The μ-transmission spaces have exact descriptions, depending on s:
On the two intervals of length 1, $J_1 = \{s \mid -\frac{1}{2} < s + a - \operatorname{Re}\delta < \frac{1}{2}\}$ and $J_2 = \{s \mid \frac{1}{2} < s + a - \operatorname{Re}\delta < \frac{3}{2}\}$, the μ-transmission spaces satisfy:

$$H^{\mu(2a+s)}(\overline{\mathbb{R}}^n_+) = \dot{H}^{2a+s}(\overline{\mathbb{R}}^n_+) \text{ when } s \in J_1,$$
$$H^{\mu(2a+s)}(\overline{\mathbb{R}}^n_+) = \dot{H}^{2a+s}(\overline{\mathbb{R}}^n_+) \dot{+} x_n^\mu K_0 H^{s'}(\mathbb{R}^{n-1}) \text{ when } s \in J_2;$$

here $s' = 2a + s - \operatorname{Re}\mu - \frac{1}{2}$, and $K_0 : \varphi \mapsto v$ is the standard Poisson operator solving $(1-\Delta)v = 0$ on $\mathbb{R}^n_+$, $\gamma_0 v = \varphi$.
Observe how the μ-th power of the distance x_n to the boundary enters when s is in the higher range.

Once we have this regularity theorem, further results can be shown. For example, the product and regularity formulas allow proving the following integration by parts formula, important for uniqueness questions:

Theorem 6. *For $u \in x_n^{\mu}\mathcal{S}(\overline{\mathbb{R}}^n_+)$, $u' \in x_n^{\bar\mu'}\mathcal{S}(\overline{\mathbb{R}}^n_+)$, there holds:*

$$\int_{\mathbb{R}^n_+} Pu\,\partial_n \bar u'\,dx + \int_{\mathbb{R}^n_+} \partial_n u\,\overline{P^* u'}\,dx \qquad (6)$$
$$= \Gamma(\mu+1)\Gamma(\mu'+1)\int_{\mathbb{R}^{n-1}} s_0\gamma_0(u/x_n^{\mu})\,\gamma_0(\bar u'/x_n^{\mu'})\,dx',$$

where $s_0 = e^{-i\pi\delta}p(0,1)$.
The formula extends to $u \in H^{\mu(t)}(\overline{\mathbb{R}}^n_+)$, $u' \in H^{\bar\mu'(t')}(\overline{\mathbb{R}}^n_+)$, when $t > \operatorname{Re}\mu + \frac12$, $t' > \operatorname{Re}\mu' + \frac12$, with $t, t' \geq a$.

The proof starts with treating $\widehat{P}$, using the factorization $r^+\widehat{P} = (r^+\Xi_-^{\mu'}e^+)(r^+\widehat{Q}e^+)\Xi_+^{\mu}$ and showing integration formulas for each factor. Then r^+P is treated by use of the fact that $P - \widehat{P}$ is of lower order.

Published in Math. in Engineering, G '22. For operators of the form $L = L_e + L_o$ studied by Dipierro, Ros-Oton, Serra and Valcinoci '20, Theorem 6 gives an alternative proof of their fundamental Prop. 1.4, based on writing $L = L^{\frac12}L^{\frac12}$ and doing real calculations.

4. Further perspectives

This is as far as this type of results for general strongly elliptic operators have been developed; they still need to be extended to curved domains. A possible strategy would be to apply more techniques from the theory of pseudodifferential operators with x-dependent symbols $p(x,\xi)$.

If one reduces the curved situation to the situation of $\mathbb{R}^n_+$ by a change of variables, the resulting operator P will have an x-dependent symbol, whose value at $(x',0)$ in the boundary of $\mathbb{R}^n_+$ depends on x'. Then also the value μ for which the principal symbol $p_0(x',0,\xi)$ can be factorized as in the discussion we have presented, will depend on x'.

One will then have to work with operators $\Xi_\pm^{\mu(x')}$ of variable order. We have a sketch of what may be done, losing an epsilon in the order of the entering Sobolev spaces.

We also expect that the localization methods given in the paper of Dipierro, Ros-Oton, Serra and Valdinoci '20 could be helpful for these more general operators.

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THANKS FOR YOUR ATTENTION!