Magnetic Pseudodifferential Super Operators

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Outline

This talk is partially based on the joint work (arXiv:2201.11487) with M. Lein (Tohoku University, Sendai, Japan).

- Motivation
- THe Magnetic Weyl Calculus
- Magnetic Pseudodifferential Super Operators
- Asymptotic Expansion
- Future Projects

Motivation

- Let H be a Hamiltonian affiliated to a von Neumann algebra $\mathfrak{A}$ endowed with a trace $\mathcal{T}$. In quantum mechanics, density operators evolve according to the Liouville equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = L_H\big(\rho(t)\big) := -\mathrm{i}[H, \rho(t)], \qquad \rho(t_0) \in \mathcal{L}^1(\mathfrak{A}).$$

- L_H maps operators to operators; physicists refer to those as super operators.
- This kind of algebraic approach was useful in many cases, e.g.,
  - systems from statistical mechanics in the thermodynamic limit [Bratelli-Robinson].
  - linear response theory [De Nittis-Lein 2017].
- Algebraic approach
  - Makes the mathematical descriptions for various systems rigorous.
  - But it involves technical assumptions on operators which seem difficult to verify for concrete models.

Motivation

- Pseudodifferential theory
  - The way of assigning functions (called symbols) to operators.
  - Translates properties of symbols to associated pseudodifferential operators.
- Question: how would technical assumptions in the algebraic approach be simplified if we regard $\rho(t)$ and L_H as pseudodifferential operators?
- We can incorporate $\rho(t)$ in the pseudodifferential theory (the Weyl calculus). But what about L_H or other super operators?

Goal

The goal of this talk is to introduce the newly constructed pseudodifferential calculus for super operators, which is a natural receptacle of L_H and other relevant super operators.

- We expect that our construction will lift the technical assumptions on operators in the algebraic approach.

The Magnetic Setting

- We consider a charged particle moving in $\mathbb{R}^d$ subjected to the magnetic field B. Pick a vector potential A so that $B = \mathrm{d}A$.
- We assume one of the following two assumptions on the magnetic field B and associated vector potential A.
  - (PB) All components of B belong to $\mathcal{C}^\infty_{\mathrm{u,pol}}(\mathbb{R}^d)$ and all components of A belong to $\mathcal{C}^\infty_{\mathrm{pol}}(\mathbb{R}^d)$.
  - (B) All components of B belong to $\mathcal{C}^\infty_{\mathrm{b}}(\mathbb{R}^d)$ and all components of A belong to $\mathcal{C}^\infty_{\mathrm{pol}}(\mathbb{R}^d)$.
- $\Xi = T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^{d*}$ (the phase space)
- Elements of Ξ will be denoted by $X = (x, \xi)$, $Y = (y, \eta)$, $Z = (z, \zeta)$ with space components $x, y, z \in \mathbb{R}^d$ and momentum components $\xi, \eta, \zeta \in \mathbb{R}^d$.
- Ξ is endowed with the symplectic form $\sigma(X, Y) := \xi \cdot y - x \cdot \eta$.

The Magnetic Weyl System

- We introduce two small parameters [Lein, JMP 2010]:
  - The coupling of the charge to the magnetic field λ.
  - A semiclassical parameter ε.

  These are crucial for asymptotic expansions arise in the study of semiclassical limits.

- In the magnetic setting, the building block observables are the position operators $Q = (Q_1, \ldots, Q_d)$ and the kinetic momentum operators $P^A = (P^A_1, \ldots, P^A_d)$ satisfying the commutation relations:

$$\mathrm{i}[Q_j, Q_k] = 0, \qquad \mathrm{i}[P^A_j, P^A_k] = \varepsilon\lambda B_{jk}(Q), \qquad \mathrm{i}[P^A_j, Q_k] = \varepsilon\delta_{jk}.$$

- It is convenient to encode these commutation relations into the magnetic Weyl system $\{w^A(X)\}_{X\in\Xi}$ defined by

$$w^A(X) := \mathrm{e}^{-\mathrm{i}\sigma(X,(Q,P^A))} = \mathrm{e}^{-\mathrm{i}(\xi\cdot Q - x\cdot P^A)}.$$

Magnetic Weyl Quantization

Definition (Magnetic Weyl Quantization [Măntoiu-Purice, JMP 2004])

For all $f \in \mathcal{S}(\Xi)$ and magnetic fields satisfying (PB), we define

$$\begin{aligned} \mathrm{op}^A(f) &:= \frac{1}{(2\pi)^d} \int_\Xi \mathrm{d}X \, (\mathcal{F}_\sigma f)(X) \, w^A(X) \\ &= \frac{1}{(2\pi)^d} \int_\Xi \mathrm{d}X \, (\mathcal{F}_\sigma f)(X) \, \mathrm{e}^{+\mathrm{i}\sigma((Q,P^A),X)}. \end{aligned}$$

Magnetic Weyl quantization inherits gauge-covariance from the magnetic Weyl system.

$$\mathrm{op}^{A+\mathrm{d}\chi}(f) = \mathrm{e}^{+\mathrm{i}\lambda\chi(Q)} \, \mathrm{op}^A(f) \, \mathrm{e}^{-\mathrm{i}\lambda\chi(Q)}.$$

The Magnetic Weyl Product

Given $f, g \in \mathcal{S}(\Xi)$, we define

$$(f \star^B g)(X) := \frac{1}{(2\pi)^{2d}} \int_\Xi \mathrm{d}Y \int_\Xi \mathrm{d}Z \, \mathrm{e}^{+\mathrm{i}\sigma(X,Y+Z)} \, e^{+\mathrm{i}\frac{\varepsilon}{2}\sigma(Y,Z)}.$$
$$\mathrm{e}^{-\mathrm{i}\frac{\lambda}{\varepsilon}\Gamma^B(\langle x-\frac{\varepsilon}{2}(y+z), x+\frac{\varepsilon}{2}(y-z), x+\frac{\varepsilon}{2}(y+z)\rangle)} \, (\mathcal{F}_\sigma f)(Y) \, (\mathcal{F}_\sigma g)(Z).$$

Then we have $\mathrm{op}^A(f)\,\mathrm{op}^A(g) = \mathrm{op}^A(f \star^B g)$.
The class of Hörmander symbols of order $m \in \mathbb{R}$ and type (ρ, δ) with $0 \leq \rho \leq \delta \leq 1$ is the Fréchet space defined by

$$S^m_{\rho,\delta}(\Xi) := \left\{ f \in \mathcal{C}^\infty(\Xi) \mid \sup_{X \in \Xi} \langle \xi \rangle^{-m-|a|\delta+|\alpha|\rho} \left| \partial_x^a \partial_\xi^\alpha f(X) \right| < \infty \; \forall a, \alpha \in \mathbb{N}_0^d \right\}.$$

Theorem ([Iftimie-Măntoiu-Purice, PRIMS 2007])

Let $m_1, m_2 \in \mathbb{R}$. For magnetic fields B satisfying (B) and $0 \leq \delta \leq \rho \leq 1$, the magnetic Weyl product $\star^B$ gives rise to a continuous bilinear map,

$$\star^B : S^{m_1}_{\rho,\delta}(\Xi) \times S^{m_2}_{\rho,\delta}(\Xi) \longrightarrow S^{m_1+m_2}_{\rho,\delta}(\Xi).$$

Asymptotic Expansion

Furthermore, we can asymptotically expand the product $f \star^B g$ in ε and λ:

Theorem ([Lein, JMP 2010])

Let $m_1, m_2 \in \mathbb{R}$ and assume B satisfies (B), $f \in S^{m_1}_{\rho,0}(\Xi)$ and $g \in S^{m_2}_{\rho,0}(\Xi)$. Then there is $N \in \mathbb{N}_0$ such that

$$f \star^B g = \sum_{l=0}^{N} \sum_{n+k=l} \varepsilon^n \lambda^k (f \star^B g)_{(n,k)} + \tilde{R}_N,$$

where $(f \star^B g)_{(n,k)} \in S^{m_1+m_2-(n+k)\rho}_{\rho,0}(\Xi)$, $\tilde{R}_N \in S^{m_1+m_2-(N+1)\rho}_{\rho,0}(\Xi)$ and the semi-norms of $\tilde{R}_N$ is sufficiently small.

Magnetic Pseudodifferential Super Operators

Some notations:

- $\Xi^2 := \Xi \times \Xi$ (the doubled phase space).
- Elements of Ξ^2 will be denoted by $\mathbf{X} = (X_L, X_R), \mathbf{Y} = (Y_L, Y_R), \mathbf{Z} = (Z_L, Z_R)$. Here we follow the same convention as before for variables in Ξ, e.g., $X_L = (x_L, \xi_L)$ and $Y_R = (y_R, \eta_R)$ with $x_L, y_R \in \mathbb{R}^d$ and $\xi_L, \eta_R \in \mathbb{R}^{d^*}$.
- We endow Ξ^2 with the symplectic form Σ defined by

$$\Sigma(\mathbf{X}, \mathbf{Y}) := \sigma(X_L, Y_L) + \sigma(X_R, Y_R).$$

- The symplectic Fourier transform $\mathcal{F}_\Sigma F$ of a function F on Ξ^2 is defined by

$$(\mathcal{F}_\Sigma)(\mathbf{X}) := \frac{1}{(2\pi)^{2d}} \int_{\Xi^2} \mathrm{d}\mathbf{X}' \, \mathrm{e}^{+\mathrm{i}\Sigma(\mathbf{X}, \mathbf{X}')} \, F(\mathbf{X}').$$

Magnetic Pseudodifferential Super Operators

Given a function $g \in \mathcal{S}(\Xi)$, set $\hat{g}^A := \mathrm{op}^A(g)$.

Definition (Magnetic Super Weyl System)

For $\mathbf{X} \in \Xi^2$, we define

$$W^A(\mathbf{X})\, \hat{g}^A \equiv W^A(X_L, X_R)\, \hat{g}^A := w^A(X_L)\ \hat{g}^A\, w^A(X_R).$$

Definition (Magnetic Pseudodifferential Super Operator)

The magnetic pseudodifferential super operator $\mathrm{Op}^A(F)$ associated with a function $F \in \mathcal{S}(\Xi^2)$ is defined by

$$\mathrm{Op}^A(F)\, \hat{g}^A := \frac{1}{(2\pi)^{2d}} \int_{\Xi^2} \mathrm{d}\mathbf{X}\, (\mathcal{F}_\Sigma F)(\mathbf{X})\, W^A(\mathbf{X})\, \hat{g}^A.$$

Magnetic Pseudodifferential Super Operators

If F is a function of the form $F(\mathbf{X}) = f_L(X_L) f_R(X_R)$, $f_L, f_R \in \mathcal{S}(\Xi)$, then we get

$$\begin{aligned}\mathrm{Op}^A(F)\,\hat{g}^A &= \frac{1}{(2\pi)^{2d}} \int_\Xi \mathrm{d}X_L \int_\Xi \mathrm{d}X_R\, (\mathcal{F}_\sigma f_L)(X_L)\, (\mathcal{F}_\sigma f_R)(X_R) \cdot \\ &\qquad w^A(X_L)\, \hat{g}^A\, w^A(X_R) \\ &= \mathrm{op}^A(f_L)\, \hat{g}^A\, \mathrm{op}^A(f_R).\end{aligned}$$

- Example: $L_h(X_L, X_R) := -\mathrm{i}h(X_L) + \mathrm{i}h(X_R)$ is the symbol of the Liouville super operator $\hat{g}^A \mapsto -\mathrm{i}[\hat{h}^A, \hat{g}^A] = -\mathrm{i}\hat{h}^A\hat{g}^A + \mathrm{i}\hat{g}^A\hat{h}^A$.

Proposition (L.-Lein)

Suppose that B satisfies (PB) and let $F \in \mathcal{S}(\Xi^2)$. Then

(1) $\mathrm{Op}^A(F)$ *induces a bounded linear operator from* $\mathcal{B}(L^2(\mathbb{R}^d))$ *to itself.*

(2) *For every* $p \geq 1$, $\mathrm{Op}^A(F)$ *gives rise to a bounded linear operator from* $\mathfrak{L}^p(\mathcal{B}(L^2(\mathbb{R}^d)))$ *to itself.*

Proof: If $\hat{g}^A \in \mathfrak{L}^p(\mathcal{B}(L^2(\mathbb{R}^d)))$, then we have $W^A(\mathbf{X})\,\hat{g}^A = w^A(X_L)\,\hat{g}^A\,w^A(X_R) \in \mathfrak{L}^p(\mathcal{B}(L^2(\mathbb{R}^d)))$ and

$$\left\| W^A(\mathbf{X})\,\hat{g}^A \right\|_p \leq \|w^A(X_L)\|\,\|\hat{g}^A\|_p\,\|w^A(X_R)\| = \|\hat{g}^A\|_p.$$

Now (2) can be proved as follows. (1) is proved by replacing $\|\cdot\|_p$ by $\|\cdot\|$.

$$\begin{aligned}\left\| \mathrm{Op}^A(f)\,\hat{g}^A \right\|_p &\leq \frac{1}{(2\pi)^{2d}} \int_{\Xi^2} \mathrm{d}\mathbf{X}\,\|(\mathcal{F}_\Sigma F)(\mathbf{X})\,W^A(\mathbf{X})\,\hat{g}^A\|_p \\ &\leq \left(\frac{1}{(2\pi)^{2d}} \int_{\Xi^2} \mathrm{d}\mathbf{X}\,|(\mathcal{F}_\Sigma F)(\mathbf{X})| \right) \|\hat{g}^A\|_p.\end{aligned}$$

Given $F \in \mathcal{S}(\Xi^2)$ and $g \in \mathcal{S}(\Xi)$, can $\mathrm{Op}^A(F)\,\hat{g}^A = \mathrm{Op}^A(F)\,\mathrm{op}^A(g)$ be seen as a magnetic Weyl pseudodifferential operator? If so, then what is its symbol? Answer:

$$F \bullet^B g(X) := \frac{1}{(2\pi)^{3d}} \int_{\Xi^2} \mathrm{d}\mathbf{Y} \int_{\Xi} \mathrm{d}Z \, \mathrm{e}^{+\mathrm{i}\sigma(X, Y_L + Y_R + Z)} \, \mathrm{e}^{+\mathrm{i}\frac{\varepsilon}{2}\sigma(Y_L + Z, Y_R + Z)} \, \mathrm{e}^{-\mathrm{i}\lambda\Omega^B(x, y_L, y_R, z)} \, (\mathcal{F}_\Sigma F)(\mathbf{Y}) \, (\mathcal{F}_\sigma g)(Z).$$

Proposition (L.-Lein)

Assume B satisfies (PB). Then the following holds.

1. $F \bullet^B g \in \mathcal{S}(\Xi)$ *and* $\mathrm{Op}^A(F)\,\mathrm{op}^A(g) = \mathrm{op}^A(F \bullet^B g)$.
2. *For the special case* $F(\mathbf{X}) = f_L(X_L) f_R(X_R)$, $f_L, f_R \in \mathcal{S}(\Xi)$, *the semi-super product reduces to*

$$F \bullet^B g = f_L \star^B g \star^B f_R.$$

The Magnetic Super Weyl Product

Given $F, G \in \mathcal{S}(\Xi^2)$, can $\operatorname{Op}^A(F)\operatorname{Op}^A(G)$ be seen as a magnetic Weyl pseudodifferential super operator? If so, then what is its symbol? Answer:

$$F \sharp^B G(\mathbf{X}) := \frac{1}{(2\pi)^{4d}} \int_{\Xi^2} \mathrm{d}\mathbf{Y} \int_{\Xi^2} \mathrm{d}\mathbf{Z}\, \mathrm{e}^{+\mathrm{i}\Sigma(\mathbf{X},\mathbf{Y}+\mathbf{Z})}\, \mathrm{e}^{+\mathrm{i}\frac{\varepsilon}{2}\Sigma(r(\mathbf{Y}),\mathbf{Z})}.$$
$$\mathrm{e}^{-\mathrm{i}\lambda\gamma^B(x_L,y_L,z_L)}\, \mathrm{e}^{-\mathrm{i}\lambda\gamma^B(x_R,z_R,y_R)}\, (\mathcal{F}_\Sigma F)(\mathbf{Y})\, (\mathcal{F}_\Sigma G)(\mathbf{Z}).$$

Here we have set $r(\mathbf{Y}) \equiv r(Y_L, Y_R) := (-Y_L, Y_R)$.

Proposition (L.-Lein)

Assume B satisfies (PB). Then we have $F \sharp^B G \in \mathcal{S}(\Xi^2)$ and $\operatorname{Op}^A(F)\operatorname{Op}^A(G) = \operatorname{Op}^A(F \sharp^B G)$.

Extension of the Calculus by Duality

Lemma

Assume B satisfies (PB) and let $F \in \mathcal{S}(\Xi^2)$. Then the map $\mathcal{S}(\Xi) \ni g \mapsto F \bullet^B g \in \mathcal{S}(\Xi)$ can be seen as the integral operator associated with the kernel,

$$K_F^B(X,Z) := \frac{\mathrm{e}^{+\mathrm{i}\frac{2}{\varepsilon}(x\cdot\xi - z\cdot\zeta)}}{(\pi\varepsilon)^{3d}} \int_{\Xi^2} \mathrm{d}\mathbf{Y}\, \mathrm{e}^{+\mathrm{i}\frac{2}{\varepsilon} y_R\cdot(\zeta-\xi)}\, \mathrm{e}^{+\mathrm{i}\frac{2}{\varepsilon} y_L\cdot(\xi+\zeta)}\,.$$
$$\mathrm{e}^{-\mathrm{i}\frac{2}{\varepsilon} y_L\cdot\eta_L}\, \mathrm{e}^{-\mathrm{i}\frac{2}{\varepsilon}(y_L - z + x)\cdot\eta_R}\, \mathrm{e}^{-\mathrm{i}\lambda\Omega^B\left(x, \frac{2}{\varepsilon}y_L, \frac{2}{\varepsilon}(x+y_L-z), \frac{2}{\varepsilon}(z-y_L-y_R)\right)}\,.$$
$$F(y_R, \eta_L, x+z-y_R, \eta_R).$$

Furthermore, the map $F \mapsto K_F^B$ gives rise to a topological vector space isomorphism from $\mathcal{S}(\Xi^2)$ to itself.

Remark

$F \mapsto K_F^B$ extends to an isomorphism $\mathcal{S}'(\Xi^2) \to \mathcal{S}'(\Xi^2)$.

Extension of the Calculus by Duality

- We can also check that, for all $F \in \mathcal{S}(\Xi^2)$ and $g, h \in \mathcal{S}(\Xi)$, we have
$$(F \bullet^B g, h)_{\mathcal{S}(\Xi)} = (g, F^t \bullet^B h)_{\mathcal{S}(\Xi)} = (K_F^B, h \otimes g)_{\mathcal{S}(\Xi^2)},$$
where we have set $F^t(X_L, X_R) := F(X_R, X_L)$, $X_L, X_R \in \Xi$.
- Given $F \in \mathcal{S}'(\Xi^2)$ and $g \in \mathcal{S}(\Xi)$, we can define $F \bullet^B g \in \mathcal{S}'(\Xi)$ by
$$(F \bullet^B g, h)_{\mathcal{S}(\Xi)} := (K_F^B, h \otimes g)_{\mathcal{S}(\Xi^2)} \qquad \forall h \in \mathcal{S}(\Xi).$$

Definition (Magnetic Semi-super Moyal Space)

We say that $F \in \mathcal{S}'(\Xi^2)$ belongs to the magnetic semi-super Moyal space $\mathfrak{m}^B(\Xi^2) \subset \mathcal{S}(\Xi^2)$ if
$$g \longmapsto F^t \bullet^B g,$$
induces a continuous linear map from $\mathcal{S}(\Xi)$ to itself.

- Given $F \in \mathfrak{m}^B(\Xi^2)$ and $g \in \mathcal{S}'(\Xi)$, we define $F \bullet^B g \in \mathcal{S}'(\Xi)$ by
$$(F \bullet^B g, h)_{\mathcal{S}(\Xi)} := (g, F^t \bullet^B h)_{\mathcal{S}(\Xi)} \qquad \forall h \in \mathcal{S}(\Xi).$$

Hörmander Super Symbol Classes

Definition (Hörmander Super Symbol Classes $S^{m_L,m_R}_{\rho,\delta}(\Xi^2)$)

Let $m_L, m_R \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta < 1$. $S^{m_L,m_R}_{\rho,\delta}(\Xi^2)$ is the Fréchet space consists of functions $F \in \mathcal{C}^\infty(\Xi^2)$ such that, for all $a_L, a_R, \alpha_L, \alpha_R \in \mathbb{N}_0^d$, there exists $C_{a_L a_R \alpha_L \alpha_R} > 0$ such that, for all $\mathbf{X} = (X_L, X_R) \in \Xi^2$, we have

$$\left|\partial_{x_L}^{a_L}\partial_{\xi_L}^{\alpha_L}\partial_{x_R}^{a_R}\partial_{\xi_R}^{\alpha_R} F(\mathbf{X})\right| \le C_{a_L a_R \alpha_L \alpha_R} \langle \xi_L \rangle^{m_L - |\alpha_L|\rho + |a_L|\delta} \langle \xi_R \rangle^{m_R - |\alpha_R|\rho + |a_R|\delta}.$$

Definition (Hörmander Super Symbol Classes $S^{m}_{\rho,\delta}(\Xi^2)$)

Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$ and $\delta < 1$. $S^{m}_{\rho,\delta}(\Xi^2)$ is the Fréchet space consists of functions $F \in \mathcal{C}^\infty(\Xi^2)$ such that, for all $a_L, a_R, \alpha_L, \alpha_R \in \mathbb{N}_0^d$, there exists $C_{a_L a_R \alpha_L \alpha_R} > 0$ such that, for all $\mathbf{X} = (X_L, X_R) \in \Xi^2$, we have

$$\left|\partial_{x_L}^{a_L}\partial_{\xi_L}^{\alpha_L}\partial_{x_R}^{a_R}\partial_{\xi_R}^{\alpha_R} F(\mathbf{X})\right| \le C_{a_L a_R \alpha_L \alpha_R} \langle (\xi_L, \xi_R) \rangle^{m - (|\alpha_L| + |\alpha_R|)\rho + (|a_L| + |a_R|)\delta}.$$

The Semi-super Product of Hörmander Symbols

Using oscillatory integral techniques, we can prove the following results. Here we assume B satisfies (B), $0 \leq \rho \leq 1$ and $0 < \varepsilon, \lambda \leq 1$.

Lemma

(1) *For any $m \in \mathbb{R}$, we have $S^m_{\rho,0}(\Xi^2) \subset \mathfrak{m}^B(\Xi^2)$.*

(2) *For any $m_L, m_R \in \mathbb{R}$, we have $S^{m_L,m_R}_{\rho,0}(\Xi^2) \subset \mathfrak{m}^B(\Xi^2)$.*

Proposition (L.-Lein)

The map $(F, g) \mapsto F \bullet^B g$ gives rise to continuous bilinear maps,

$$\bullet^B : S^m_{\rho,0}(\Xi^2) \times S^{m'}_{\rho,0}(\Xi) \longrightarrow S^{m+m'}_{\rho,0}(\Xi)$$

$$\bullet^B : S^{m_L,m_R}_{\rho,0}(\Xi^2) \times S^m_{\rho,0}(\Xi) \longrightarrow S^{m+m_L+m_R}_{\rho,0}(\Xi).$$

The Super Product of Hörmander Symbols

By using the similar duality technique, we can also define the subspace $\mathcal{M}^B(\Xi^2)$ of $\mathcal{S}'(\Xi^2)$ such that $F\sharp^B G$ makes sense for all $F, G \in \mathcal{M}^B(\Xi^2)$. Again, by applying oscillatory integral techniques, we get the following results. Again, we assume B satisfies (B), $0 \le \rho \le 1$ and $0 < \varepsilon, \lambda \le 1$.

Lemma

(1) *For any $m \in \mathbb{R}$, we have $S^m_{\rho,0}(\Xi^2) \subset \mathcal{M}^B(\Xi^2)$.*

(2) *For any $m_L, m_R \in \mathbb{R}$, we have $S^{m_L,m_R}_{\rho,0}(\Xi^2) \subset \mathcal{M}^B(\Xi^2)$.*

Proposition (L.-Lein)

The map $(F, G) \mapsto F\sharp^B G$ gives rise to continuous bilinear maps,

$$\sharp^B : S^m_{\rho,0}(\Xi^2) \times S^{m'}_{\rho,0}(\Xi^2) \longrightarrow S^{m+m'}_{\rho,0}(\Xi^2)$$

$$\sharp^B : S^{m_L,m_R}_{\rho,0}(\Xi^2) \times S^{m'_L,m'_R}_{\rho,0}(\Xi^2) \longrightarrow S^{m_L+m'_L,m_R+m'_R}_{\rho,0}(\Xi^2).$$

Asymptotic Expansion

Both $\bullet^B$ and $\sharp^B$ can be asymptotically expanded in ε and λ:

Theorem (L.-Lein)

Assume B satisfies (B). Then the following holds.

(1) *Let $F \in S^{m}_{\rho,0}(\Xi^2)$ and $g \in S^{m'}_{\rho,0}(\Xi)$. Then there is $N \in \mathbb{N}_0$ such that*

$$F \bullet^B g = \sum_{l=0}^{N} \sum_{n+k=l} \varepsilon^n \lambda^k (F \bullet^B g)_{(n,k)} + \tilde{R}_N, \qquad (*)$$

where $(F \bullet^B g)_{(n,k)} \in S^{m+m'-\rho(n+k)}_{\rho,0}(\Xi)$, $\tilde{R}_N \in S^{m+m'-\rho(N+1)}_{\rho,0}(\Xi)$ and the semi-norms of $\tilde{R}_N$ is sufficiently small.

(2) *If $F \in S^{m_L,m_R}_{\rho,0}(\Xi^2)$ and $g \in S^{m}_{\rho,0}(\Xi)$, then we also have an asymptotic expansion of $F \bullet^B g$ as in (*). In this case, $(F \bullet^B g)_{(n,k)} \in S^{m+m_L+m_R-\rho(n+k)}_{\rho,0}(\Xi)$, $\tilde{R}_N \in S^{m+m_L+m_R-\rho(N+1)}_{\rho,0}(\Xi)$ and the semi-norms of $\tilde{R}_N$ is sufficiently small.*

Asymptotic Expansion

Theorem (L.-Lein)

Assume B satisfies (B) and let $F \in S^{m}_{\rho,0}(\Xi^2)$ and $G \in S^{m'}_{\rho,0}(\Xi^2)$. Then there is $N \in \mathbb{N}_0$ such that

$$F \sharp^B G = \sum_{l=0}^{N} \sum_{n+k=l} \varepsilon^n \lambda^k (F \sharp^B G)_{(n,k)} + \tilde{R}_N,$$

where $(F \sharp^B G)_{(n,k)} \in S^{m+m'-\rho(n+k)}_{\rho,0}(\Xi^2)$, $\tilde{R}_N \in S^{m+m'-\rho(N+1)}_{\rho,0}(\Xi^2)$ and the semi-norm of $\tilde{R}_N$ is sufficiently small.

Future Projects

- Commutator criteria:
  - There is the criterion using commutators to determine whether a given operator is a magnetic pseudodifferential opeator [Iftimie-Măntoiu-Purice, CPDE 2010].
  - Can we establish a similar criterion to determine whether a given super operator is pseudodifferential?

Future Projects

- Link to the algebraic setting:
  - Given a symbol $f(X) = f(x, \xi)$, the behavior of the function $x \mapsto f(x, \xi)$ at each $\xi \in \mathbb{R}^{d^*}$ encode properties of a given physical system such as periodicity, disorder or decaying property. This can be taken into account by confine the coefficient to some C^*-algebra. It can be shown that the algebra of pseudodifferential operator with coefficients in a C^*-algebra is isomorphic to a twisted crossed product algebra (see, e.g., [Lein-Măntoiu-Richard, PRIMS 2010]).
  - Relation of the super operator calculus to the algebraic setting described above? For example, what would be the appropriate mapping property on noncommutative $\mathfrak{L}^p$-spaces?

Thank you for your attention!

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