Surface Waves as Fourier Distributions with Complex Phase

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The Initial Boundary Value Problem (IBVP)

Let L be a first order hyperbolic operator with constant multiplicities. Consider the general IBVP

$$\begin{cases} Lu = \partial_t u + A_0 \partial_x u + \displaystyle\sum_{j=1}^{d} A_j \partial_{y_j} u = f(t,x,y) & \text{in} \quad (0,T) \times \mathbb{R}^{1+d}_+, \\ Bu = g(t,y) & \text{on} \quad (0,T) \times \mathbb{R}^d, \\ u|_{t=0} = u_0(x,y) & \text{in} \quad \mathbb{R}^{d+1}_+, \end{cases} \tag{1}$$

where

$$\mathbb{R}^{1+d}_+ := \{(x,y) \in \mathbb{R}^{1+d} \colon x > 0,\ y \in \mathbb{R}^d\},$$

and the matrices $A_j \in M_{n\times n}(\mathbb{C})$ and $B \in M_{p\times n}(\mathbb{C})$ have constant entries.

Assume that the boundary is non-characteristic, and let p be the number of incoming characteristics, that is the number of negative eigenvalues of A_0.

The Kreiss-Lopatinskii condition

Let $\mathcal{A}(\tau, \eta)$ be the resolvent obtained after taking the Laplace-Fourier transform in the first equation of problem (1),

$$\mathcal{A}(\tau, \eta) = -(A_0)^{-1} \left(\tau I_n + i \sum_{j=1}^{d} A_j \eta_j \right), \quad \tau = \gamma + i\rho,\ \gamma \geq 0, \eta \in \mathbb{R}^d$$

For $\Re\tau > 0$, hyperbolicity (via Hersh's lemma) allow us to decompose

$$\mathbb{C}^N = E_-(\tau, \eta) \bigoplus E_+(\tau, \eta),$$

where $E_\mp(\tau, \eta)$ denote the sum of eigenspaces corresponding to eigenvalues of $\mathcal{A}(\tau, \eta)$ with negative/positive real part, respectively.

The Kreiss-Lopatinskii Condition

Well-posedness of the IBVP requires $B : E_-(\tau, \eta) \to \mathbb{C}^p$ to be an isomophism.

Definition

We say that the hyperbolic IBVP (1) satisfy the **Kreiss-Lopatinskii condition** (KL) if, for every $\eta \in \mathbb{R}^d$ and $\Re\tau > 0$,

$$E_-(\tau, \eta) \cap \mathrm{Ker}\, B = \{0\}$$

Equivalently, for all $\Re\tau > 0$ and $\eta \in \mathbb{R}^d$, there exists a function $C(\tau, \eta) > 0$ such that

$$|V| \leq C(\tau, \eta)|BV|, \qquad \forall V \in E_-(\tau, \eta).$$

This condition implies $\mathcal{C}^\infty$ well-posedness of the IBVP. In Sobolev spaces, however, one has to expect a loss of regularity.

The Uniform Kreiss-Lopatinskii Condition

From now on, we assume (KL) to hold, since otherwise the IBVP would be ill-posed.

Definition

We say that the hyperbolic IBVP (1) satisfy the **uniform Kreiss-Lopatinskii condition** (UKL) if for every $\eta \in \mathbb{R}^d$ and $\Re\tau = 0$,

$$|V| \leq C|BV|, \qquad \forall V \in E_-(\tau, \eta).$$

For $\Re\tau = 0$, Hersh's lemma is not true anymore and $E_-(\tau, \eta)$ is defined by extension. Satisfiying the (UKL) implies that dimension of $E_-(\tau, \eta)$ remains to be p when $\Re\tau \to 0$. In this case, the IBVP is strongly well-posed, with estimates comparable to those one has for the Cauchy problem.

The Uniform Kreiss-Lopatinskii Condition

The **Lopatinskii determinat** is an effective tool to check this condition. It is a function Δ that vanishes precisely at points (τ, η) where (UKL) is violated.

Let $\{X_1(\tau, \eta), \dots, X_p(\tau, \eta)\}$ be a basis of $E_-(\tau, \eta)$, continuous for $\Re\tau \geq 0$ and holomorphic $\Re\tau > 0$. Define

$$\Delta(\tau, \eta) = \det(BX_1(\tau, \eta), \dots, BX_p(\tau, \eta)).$$

Then, the IBVP is generically *weakly* well-posed, if

$$\Delta(i\rho, \eta) = 0, \text{ for some } (\rho, \eta) \in \mathbb{R}^{1+d}.$$

In this case, the problem is C^∞ well-posed but there is a lost of regularity, which has to be work out in a case-by-case study. We are interested in those IBVP that admit **surface waves** as solutions.

Surface waves

A surface wave is a non-trivial solutions of the form

$$e^{i(\rho t+\eta y)}V(x), \qquad V \in L^2(\mathbb{R}_+),$$

which represents a wave, of finite energy, travelling in the direction parallel to the boundary.Their existence relates to the failure of the (UKL) in some elliptic point. To be precise,

- a point (τ, η), with $\Re\tau = 0$, is **elliptic** if $\mathcal{A}(\tau, \eta)$ have only purely imaginary eigenvalues.
- An IBVP admits surface waves if Δ has a simple zero at some elliptic point (ρ, η).

An example for this type of solutions are the *Rayleigh waves* known in elasticity, see for example [AH13] and [Hun11].

Linear elasticity

Let K be the half space $\{x \in \mathbb{R}^2 \colon x_1 \geq 0\}$ and consider the equation of linear elasticity in an isotropic medium subjected to inhomogeneous boundary condition,

$$\begin{aligned} Lu = \partial_t^2 u - (\lambda + \mu)\nabla(\text{div } u) - \mu\Delta u = 0 \text{ in } \mathbb{R} \times K, \\ \sum_i (n_i \sigma_{ij}) = f_j \text{ on } \mathbb{R} \times \partial K, \end{aligned} \tag{2}$$

where λ and μ are positive constants, n is normal to ∂K and the stress tensor σ is

$$\sigma_{ij} = \lambda(\text{div } u)\delta_{ij} + \mu(\partial_{x_j} u_i + \partial_{x_i} u_j).$$

We will follow the analysis in [Tay79] to better understand Rayleigh waves.

Approximated solution

It is possible to construct an approximated solution to (2) in the form

$$u = \int e^{i\phi(t,x,\zeta)} a(t,x,\zeta)\widehat{F}(\zeta)\ d\zeta + \int e^{i\psi(t,x,\zeta)} b(t,x,\zeta)\widehat{G}(\zeta)\ d\zeta,$$

where $\zeta = (\tau, \eta)$,

- ϕ and ψ are complex-value phase functions satisfying the eikonal equations

$$\phi_t = (\lambda + 2\mu)^{1/2}|\nabla_x \phi|, \qquad \psi_t = \mu^{1/2}|\nabla_x \psi|.$$

- For $x \in \partial K$, the vector-value amplitudes a and b take values in the eigenspaces of the symbol $L(t, x, \tau, \xi)$.
- F and G are scalar-valued distributions given by

$$T \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{3}$$

 where T is a pseudodifferential operator of the first order, determined by the boundary condition.

Classification of frequencies

Solving the eikonal equations, one finds that $T^*(\mathbb{R} \times \partial K)$ is divided into three regions

I. The **hyperbolic region**: $|\tau| > (\lambda + 2\mu)^{1/2}|\xi_2|$. Here both ϕ and ψ are real valued.

II. The **mixed region**: $\mu^{1/2}|\xi_2| < |\tau| < (\lambda + 2\mu)^{1/2}|\xi_2|$. Here ψ is real valued but ϕ is not.

III. The **elliptic region**: $|\tau| < \mu^{1/2}|\xi_2|$. Here both $\Im\phi$ and $\Im\psi$ are not zero.

It follows that the operator T is of real principal type, with exactly one simple zero, which belongs to the elliptic region. Hence, one can construct a parametrix for the boundary equation (3), finding F and G up to a smooth factor.

Propagation of singularities

The main result of [Tay79] is on the propagation of singularities. The key points are

- The wave front set of the Rayleigh wave is the set Σ given by the union of $S = WF(f_1) \cup WF(f_2)$ and the set of null-bicharacteristics of $\det \sigma(T)$ passing over S.
- The solution u is smooth up to the boundary ∂K, except at the projection to $\mathbb{R} \times \partial K$ of Σ.
- The null-bicharacteristics of L cannot pass over the set Σ twice, so there are no singularities being propagated to the interior.

Generalities: Ansatz

The previous construction hold for a hyperbolic problem of the form (1). In this case, the approximated solution would be given by a sum of p integrals as the ones above,

$$u = \sum_{j=1}^{p} \int e^{i\phi_j(x,z,\zeta)} a_j(x,z,\zeta) \widehat{F}_j(\zeta)\, d\zeta, \tag{4}$$

with phase functions of the form

$$\phi_j(t,x,\zeta) = z \cdot \zeta + \lambda_j(\zeta) x.$$

where $\lambda_j(\zeta)$ denote the eigenvalues of $\mathcal{A}(\tau,\eta)$ with $\Im \lambda_j \geq 0$, counted with their multiplicity.

Note that this are Fourier integral distributions with complex-valued phases, so we use the theory developed by Melin and Sjöstrand (see [MS75]).

Generalities: Amplitudes

Since the *stationary phase formula* remains valid for complex-valued phase functions, we can use the method of *geometric optics* to find the amplitude functions $a_j(x, z, \zeta)$.

For a first-order hyperbolic operator L and phase functions ϕ_j as before, one finds that

- For all j, the principal parts of a_j are independent of (x, z).
- Each $a_j(x, z, \zeta)$ belongs to the $\lambda_j(\zeta)$-eigenspace of

$$\mathcal{A}(\zeta) = \mathcal{A}(\tau, \eta).$$

  In fact, their principal parts form a basis of $E_-(\tau, \eta)$.

Generalities: The boundary operator T

Assuming the existence of surface waves, the operator T is of real principal type. Since it can always be chosen such that

$$\det \sigma^{(1)}(T)(\tau, \eta) = \Delta(\tau, \eta). \tag{5}$$

Indeed, replacing the ansatz u into the boundary condition of the IBVP (1), we see that

$$T(F) = \int e^{iz\cdot\zeta} M(z, \zeta) \widehat{F}(\zeta)\, d\zeta, \quad F = (F_1, \ldots, F_p)^T,$$

where the columns of the matrix $M \in M_{p\times p}(\mathbb{C})$ are

$$Ba_j(0, z, \zeta).$$

To satisfy (5), it is enough to take the principal parts of a_j as the basis used in the construction of Δ.

References

[1] Lawrence Austria and John K Hunter. "Nonlinear variational surface waves". In: *Communications in Information and Systems* 13.1 (2013), pp. 3–43.

[2] John K Hunter. "Nonlinear hyperbolic surface waves". In: *Nonlinear conservation laws and applications*. Springer, 2011, pp. 303–314.

[3] Anders Melin and Johannes Sjöstrand. "Fourier integral operators with complex-valued phase functions". In: *Fourier integral operators and partial differential equations*. Springer, 1975, pp. 120–223.

[4] Michael E. Taylor. "Rayleigh Waves in Linear Elasticity as a Propagation of Singularities Phenomenon". In: 1979.