Heat content and geometric analysis

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The shape of things to come

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- Definitions and notation

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Compare and contrast heat content with other geometric invariants.

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$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u \text{ on } D \times (0, \infty) \\ u(x, 0) &= 1 \text{ on } D \\ u(x, t) &= 0 \text{ on } \partial D \times (0, \infty). \end{aligned}$$

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The *heat content of D* is the function $HC(t) : (0, \infty) \to \mathbb{R}$ defined by

$$HC(t) = \int_D u(x, t) dx.$$

A little more notation

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Then

$$HC(t) = \sum_{\lambda \in \mathrm{spec}(D)} a_\lambda^2 e^{-\lambda t}$$

where

$$a_\lambda = \int_D \phi_\lambda(x) dx.$$

Properties of a_λ and heat trace

The a_λ measure "how volume is partitioned among eigenspaces."

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Let $\mathrm{spec}^*(D)$ be the Dirichlet spectrum without multiplicity. For $\mu \in \mathrm{spec}^*(D)$, let P_μ be orthogonal projection on the eigenspace given by μ. Then

$$\|P_\mu(1)\|_2^2 = \sum_{\lambda \in \mathrm{spec}(D),\ \lambda = \mu} a_\lambda^2.$$

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$$HT(t) = \sum_{\mu \in \mathrm{spec}^*(D)} M_\mu e^{-\mu t}$$

where M_μ is the trace of P_μ.

Moments of heat content

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In fact, it can be shown that

- Generically, the sequence A_n determines $\mathrm{spec}(D)$.
- The heat content moments determine $\{\mu \in \mathrm{spec}^* : \|P_\mu(1)\|_2 > 0\}$ (Colladay-Langford-M, (2018) JGA).

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The first moment of heat content is $A_1 = \int_D u_1(x)dx$.

The first moment is a classical object known as *torsional rigidity.* It appears in many geometric inequalities, eg $A_1\lambda_1 \leq \mathrm{Vol}(D)$.

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$$\|\mathbb{E}^x[\tau^n]\|_1 = A_n.$$

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Aizenman and Simon gave a rearrangement argument for all moments for domains in Euclidean space in 1982.

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There is much current interest in quantitative versions of the above (see Brasco-de Phillips (2016) arxiv).

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Recent related work: Palmer-Sarrion-Pedralva (2022) arxiv.

Torsional rigidity and comparison in a singular setting

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For compact quantum graphs it's possible to compute torsional rigidity (Colladay-Kaganovski-M (2016) J Phys A). There are recent comparison results for torsional rigidity on compact quantum graphs (see Mugnolo-Plummer (2022)).

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For planar polygons, these formulae terminate after three terms.

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For more complicated polygons, little is known.

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Among these sets are the *Chapman domains*

A pair of isospectral sets

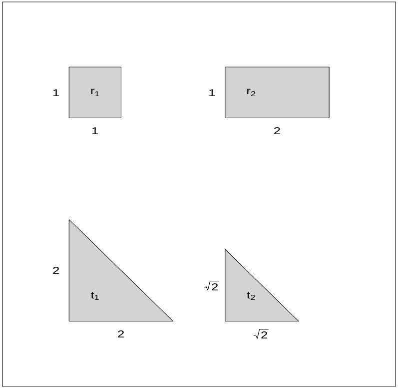

Figure: Chapman domains C_1 and C_2 are disjoint unions of a rectangle and a triangle: $C_i = r_i \cup t_i$.

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Similarly, for quantum graphs Colladay-Kaganovski-M (2016) prove:

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There are pairs of isospectral quantum graphs distinguished by torsional rigidity.

Fin

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Questions?