Index of families of transversally elliptic operators with respect to central extensions

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Family of $Spin(2n)$-transversally elliptic Dirac operators

Index of elliptic operators on closed manifolds

Let

- $E^{\pm} \to Z$ be vector bundles on a compact manifold Z
- $D : C^{\infty}(Z, E^{+}) \to C^{\infty}(Z, E^{-})$ be an order m pseudodifferential operator on Z
- $\sigma(D) \in C^{\infty}(T^{*}Z, \mathrm{Hom}(E^{+}, E^{-}))$ be the principal symbol of D.

Definition

D is elliptic if its principal symbol $\sigma(D)$ is invertible outside the 0 section on $T^{*}Z$.

$$D \text{ elliptic} \Rightarrow D \text{ Fredholm}$$

Theorem (Atiyah-Singer '63)

The index of an elliptic operator D can be computed from its principal symbol $\sigma(D)$ and only depends on its K-theory class $[\sigma(D)] \in \mathrm{K}(T^{*}Z)$. We have

$$\mathrm{Ind}(D) = \int_{T^{*}Z} \mathrm{Ch}(\sigma(D)) \wedge \mathrm{Td}(T^{*}Z \otimes \mathbb{C}).$$

Index of G-invariant elliptic operators

Let G be a compact Lie group acting on Z. Assume D elliptic and G-invariant. Then

$$\ker D \text{ and } \operatorname{coker} D \quad \text{are finite dimensional representations of } G.$$

Definition

The index of a G-invariant elliptic operator is

$$\mathrm{Ind}_G(D) = \chi_{\ker D} - \chi_{\operatorname{coker} D} \in R(G).$$

Theorem (Atiyah-Singer '68)

The index of a G-invariant elliptic operator D can be computed from its principal symbol $\sigma(D)$ and only depends on its K*-theory class $[\sigma(D)] \in \mathrm{K}_G(T^*Z)$.*

G-transversally elliptic operators

Denote by $\mathfrak{g}$ the Lie algebra of G. The G-transversal space is :

$$T^*_G Z = \{(x, \xi) \in T^* Z \mid \xi(X_Z(x)) = 0, \ \forall X \in \mathfrak{g}\}.$$

Definition

A G-invariant pseudodifferential operator D is said G-transversally elliptic if its principal symbol is invertible outside the 0 section on $T^*_G Z$.

Generally, $\ker D$ and $\operatorname{coker} D$ are not finite dimensional representations of G.

Theorem (Singer '73 - Atiyah '74)

For every unitary irreducible representation V of G, the multiplicity of V in $\ker D$ and $\operatorname{coker} D$ are finite. The index of D defines a central distribution :

$$\operatorname{Ind}_a(D) = \chi_{\ker D} - \chi_{\operatorname{coker} D} \in C^{-\infty}(G)^{Ad(G)},$$

which depends only on $[\sigma(D)] \in K_G(T^*_G Z)$. In fact, $\operatorname{Ind}_a(D)$ belongs to the Sobolev space $H_{-\dim Z}(G)$.

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Index of elliptic families

Let

- $p : Z \to B$ be a G-equivariant compact fibration, i.e. $p(gx) = gp(x)$,
- $Z_b = p^{-1}(\{b\})$, $E_b^{\pm} = E^{\pm}|_{Z_b}$,
- $T(Z|B) = \ker dp = \bigsqcup_{b \in B} TZ_b$ and $T^*(Z|B)$ be the vertical tangent and cotangent bundles,
- $U_g^{\pm} : C^{\infty}(Z_b, E_b^{\pm}) \to C^{\infty}(Z_{gb}, E_{gb}^{\pm})$, $U_g^{\pm} s(x) = g(s(g^{-1}x))$, $x \in Z_{gb}$.

Definition

A continuous family $D = \big(D_b : C^{\infty}(Z_b, E_b^+) \to C^{\infty}(Z_b, E_b^-)\big)_{b \in B}$ of pseudodifferential operators is said

- *G-invariant if $U_g^- \circ D_{g^{-1}b} \circ U_{g^{-1}}^+ = D_b$,*
- *elliptic if $\sigma(D)(x, \xi) = \sigma(D_b)(x, \xi)$ is invertible $\forall (x, \xi) \in T^*(Z|B) \setminus \{0\}$.*

Theorem (Atiyah-Singer '71)

The index of a G-invariant family $D = (D_b)_{b \in B}$ of elliptic operators

$$\mathrm{Ind}^{Z|B}(D) = [\ker D] - [\mathrm{coker}\, D] \in K_G(B)$$

can be computed from the K-theory class $[\sigma(D)] \in K_G(T^*(Z|B))$.

Families of G-transversally elliptic operators

If G acts trivially on B then the vector field X_Z is vertical

$$X_Z(x) = \frac{d}{dt}|_{t=0} e^{tX} \cdot x \in T_x(Z|B), \ \forall X \in \mathfrak{g} = \mathrm{Lie}(G),$$

because $p(gx) = p(x), \forall g \in G$. The vertical G-transversal space is

$$T^*_G(Z|B) := \{(x,\xi) \in T^*(Z|B) \mid \xi(X_Z(x)) = 0, \ \forall X \in \mathfrak{g}\}.$$

Definition

A G-invariant family $D = (D_b)_{b\in B}$ is a family of G-transversally elliptic operators if $\sigma(D)$ is invertible on $T^*_G(Z|B) \smallsetminus \{0\}$.

Theorem (B. 2018, multiplicities)

For any irreducible representation V of G, the family D^G_V defined as the restriction to G-invariant sections of $D \otimes \mathrm{Id}_V$

$$(D^G_V)_b = D_b \otimes \mathrm{Id}_V : C^\infty(Z_b, E^+_b \otimes V)^G \to C^\infty(Z_b, E^-_b \otimes V)^G$$

has a well defined index

$$m_D(V) := [\ker D^G_V] - [\mathrm{coker}\, D^G_V] \in K(B).$$

Index of families of G-transversally elliptic operators

Definition

We can define the index of a family D of G-transversally elliptic operators by

$$\mathrm{Ind}^{Z|B}(D) := \sum_{V\in\hat{G}} m_D(V)\chi_V \in \mathrm{Hom}(R(G), K(B)) \cong KK(C^*G, C(B)).$$

The Chern character $\mathrm{Ch} : K(B) \to \mathcal{H}^{ev}_{dR}(B, \mathbb{C})$ *in de Rham cohomology gives*

$$\mathrm{Ind}^{Z|B}_{-\infty}(D) := \sum_{V\in\hat{G}} \mathrm{Ch}(m_D(V))\chi_V \in \mathrm{Hom}(R(G), \mathcal{H}_{dR}(B, \mathbb{C})).$$

Proposition (B. 2018)

Assume B oriented and G-trivial. We have

$$\mathrm{Ind}^{Z|B}_{-\infty}(D) := \sum_{V\in\hat{G}} \mathrm{Ch}(m_D(V))\chi_V \in C^{-\infty}(G)^{Ad(G)} \otimes \mathcal{H}_{dR}(B, \mathbb{C}),$$

and the distributional support of $\mathrm{Ind}^{Z|B}_{-\infty}(D)$ *is contained in* $\{g \in G, Z^g \neq \emptyset\}$.

Cohomological index theorem *à la* Berline-Paradan-Vergne

Theorem (B. 2018)

Assume B oriented and G-trivial. Denote by N^s the normal vector bundle to Z^s in Z.
1. There is a unique generalized function with values in the cohomology of B denoted

$$\mathrm{Ind}_{coh}^{Z|B} : K_G(T_G^*(Z|B)) \to C^{-\infty}(G)^{Ad(G)} \otimes \mathcal{H}_{dR}^{ev}(B, \mathbb{C})$$

satisfying the following local relations :

$$\mathrm{Ind}_{coh}^{M|B}([\sigma])\|_s(Y) = (2i\pi)^{-\dim(Z^s|B)} \int_{T(Z^s|B)|B} \frac{\mathrm{Ch}_c(\sigma, r^*\omega, s)(Y) \wedge \hat{A}^2(T(Z^s|B), Y)}{D_s(N^s, Y)},$$

$\forall s \in G$, $\forall Y \in \mathfrak{g}(s)$ small enough such that $\hat{A}^2(T(Z^s|B), Y)$ and $D(N^s, Y)$ are defined.
2. Furthermore, we have the following index formula :

$$\mathrm{Ind}_{coh}^{Z|B}([\sigma]) = \mathrm{Ind}_{-\infty}^{Z|B}([\sigma]) \in C^{-\infty}(G)^{Ad(G)} \otimes \mathcal{H}_{dR}^{ev}(B, \mathbb{C}).$$

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Framework

Let $0 \to \Gamma \to \widetilde{G} \xrightarrow{\zeta} G \to 0$ be a central extension of G by a finite group Γ.

Consider a compact fibration $p : M \to B$ and a G-principal bundle $\pi : P \to M$.

We obtain

- an action of $\tilde{G}$ on P given by

$$\tilde{g} \cdot p = \zeta(\tilde{g}) \cdot p, \ \ \tilde{g} \in G, \ \ p \in P,$$

- a compact $\tilde{G}$-fibration $p \circ \pi : P \to B$, with trivial action on B,

We are interested in families of $\tilde{G}$-transversally elliptic operators

$$D = \big(D_b : C^{\infty}(P_b, E_b^+) \to C^{\infty}(P_b, E_b^-)\big)_{b \in B}.$$

First observations

Since G acts freely on P,

- $T^*_G(P|B) \cong \pi^* T^*(M|B)$ is a vector bundle, and
- quotient by G defines a G-principal bundle $T^*_G(P|B) \to T^*(M|B)$.

Since Γ is discrete, we have

- $\tilde{\mathfrak{g}} = \mathfrak{g}$,
- $T^*_{\tilde{G}}(P|B) = T^*_G(P|B)$ is a vector bundle.

Moreover, we have $\{\tilde{g} \in \tilde{G},\ P^{\tilde{g}} \neq \varnothing\} = \Gamma$ and $P^\gamma = P$, $\forall \gamma \in \Gamma$.

Therefore, if $D = \left(D_b : C^\infty(P_b, E_b^+) \to C^\infty(P_b, E_b^-)\right)_{b \in B}$ is a family of $\tilde{G}$-transversally elliptic operators then $\mathrm{Ind}^{P|B}_{-\infty}(D)$ is supported in Γ and

$$\mathrm{Ind}^{P|B}_{-\infty}(D) = \sum_{\gamma \in \Gamma} Q_\gamma(D),$$

where $Q_\gamma(D) \in C^{-\infty}(\tilde{G})^{Ad(\tilde{G})} \otimes \mathcal{H}_{dR}(B, \mathbb{C})$ is supported at $\gamma \in \Gamma$.

Distributional index of families of $\tilde{G}$-transversally elliptic operators

Let $D = \big(D_b : C^\infty(P_b, E_b^+) \to C^\infty(P_b, E_b^-)\big)_{b\in B}$ be a family of $\tilde{G}$-transversally elliptic operators. Then $\mathrm{Ind}_{-\infty}^{P|B}(D)$ is supported in Γ and therefore

$$\mathrm{Ind}_{-\infty}^{P|B}(D) = \sum_{\gamma\in\Gamma} Q_\gamma(D),$$

where $Q_\gamma(D) \in C^{-\infty}(\tilde{G})^{Ad(\tilde{G})} \otimes \mathcal{H}_{dR}(B, \mathbb{C})$ is supported at $\gamma \in \Gamma$.

Theorem (B. 2021)

For all $\varphi \in C^\infty(\tilde{G})^{Ad(\tilde{G})}$, *we have*

$$\langle Q_\gamma(D), \varphi\rangle = (2i\pi)^{-\dim(M|B)} \int_{T^*(M|B)|B} \mathrm{Ch}_\gamma(\sigma(D)) \wedge \mathrm{Td}(T^*(M|B)\otimes\mathbb{C}) \wedge \mathrm{cw}(\varphi(\gamma e^X)).$$

In particular, if $\varphi = 1$ *in a neighborhood of* γ *then*

$$\langle Q_\gamma(D), \varphi\rangle = (2i\pi)^{-\dim(M|B)} \int_{T^*(M|B)|B} \mathrm{Ch}_\gamma(\sigma(D)) \wedge \mathrm{Td}(T^*(M|B) \otimes \mathbb{C}).$$

Equivariant cohomology

Let H be a compact Lie group which acts on a manifold Z. Denote by $\mathfrak{h} = \mathrm{Lie}(H)$ and consider

$$\mathcal{A}^{\infty}_{H,c}(\mathfrak{h}, Z) := \Big(C^{\infty}(\mathfrak{h}) \otimes \mathcal{A}_c(Z)\Big)^{H},$$

with the equivariant differential

$$(d_{\mathfrak{h}}\alpha)(X) = d(\alpha(X)) - \iota(X_Z)(\alpha(X)), \ \forall X \in \mathfrak{h}.$$

Then $d_{\mathfrak{h}}^2 = 0$ on $\mathcal{A}^{\infty}_{H,c}(\mathfrak{h}, Z)$ and we denote the cohomology of the complex $(\mathcal{A}^{\infty}_{H,c}(\mathfrak{h}, Z), d_{\mathfrak{h}})$ by

$$\mathcal{H}^{\infty}_{H,c}(\mathfrak{h}, Z) := H\Big(\mathcal{A}^{\infty}_{H,c}(\mathfrak{h}, Z), d_{\mathfrak{h}}\Big).$$

Remark

Since Γ acts trivially on P, we have $\mathcal{H}^{\infty}_{G,c}(\mathfrak{g}, T^*_G(P|B)) = \mathcal{H}^{\infty}_{\tilde{G},c}(\mathfrak{g}, T^*_{\tilde{G}}(P|B))$.

Equivariant Chern character (Berline-Vergne)

Let $\gamma \in \Gamma$ and $E \to T^*_G(P|B)$ a $\tilde{G}$-equivariant vector bundle then

- $P^\gamma = P$,
- γ^E linear action on the fibers of the bundle, i.e. $\gamma^E_\xi : E_\xi \to E_\xi$, $\xi \in T^*_G(P|B)$.

Definition (γ-equivariant Chern character of a connection)

Let $\gamma \in \Gamma$. If ∇ is a $\tilde{G}$-invariant connection on $E \to T^*_G(P|B)$ then

$$\mathrm{Ch}^{\tilde{G}}_\gamma(\nabla)(X) = \mathrm{Tr}(\gamma^E e^{F+\mu^E(X)}),$$

where

- $F = \nabla^2$ *is the curvature of* ∇,
- $\mu^E(X) = \mathcal{L}^E(X) - \nabla_X$ *is the moment of* ∇,
- $\mathcal{L}^E(X)s(x) = \frac{d}{dt}_{|t=0}(U_{e^{tX}}s)(x) = \frac{d}{dt}_{|t=0} e^{tX}(s(e^{-tX}x))$ *is the Lie derivative on* E.

Equivariant Chern character of symbols

- $D = \left(D_b : C^\infty(P_b, E_b^+) \to C^\infty(P_b, E_b^-)\right)_{b\in B}$ family of $\tilde{G}$-transversally elliptic operators
- $\Pi : T^*_G(P|B) \to P$ the projection,
- ∇^+ a $\tilde{G}$-invariant connection on $E^+ \to P$,
- $\nabla^{\Pi^* E^+} := \Pi^*\nabla^+$.

Since $\mathrm{supp}(\sigma(D)) \cap T^*_G(P|B) \subset M$ is compact, we can define on $\Pi^* E^- \to T^*_G(P|B)$ a connection $\nabla^{\Pi^* E^-}$ such that the relation

$$\nabla^{\Pi^* E^-} = \sigma(D) \circ \nabla^{\Pi^* E^+} \circ \sigma(D)^{-1}$$

holds outside a compact subset of $T^*_G(P|B)$.

Definition (Berline-Paradan-Vergne)

The γ-equivariant Chern character with compact support of $\sigma(D)$ is

$$\mathrm{Ch}^{\tilde{G}}_\gamma(\sigma(D))(X) := \mathrm{Ch}^{\tilde{G}}_\gamma(\nabla^{\Pi^* E^+})(X) - \mathrm{Ch}^{\tilde{G}}_\gamma(\nabla^{\Pi^* E^-})(X) \in \mathcal{H}^\infty_{G,c}(\mathfrak{g}, T^*_G(P|B)).$$

Chern-Weil (iso)morphism

- $E_1, \cdots, E_r$ basis of $\mathfrak{g}$,
- $\theta = \sum \theta_i \otimes E_i \in (\mathcal{A}^1(P) \otimes \mathfrak{g})^G$ connection 1-form on $P \to M$,
- $\mathcal{A}^{ev}_{hor}(P)$ algebra of horizontal forms of even degree on P,
- $S(\mathfrak{g})^{Ad(G)}$ ring of $Ad(G)$-invariant polynomial function on $\mathfrak{g}$.

Denote by $\Theta = \sum \Theta_i \otimes E_i \in (\mathcal{A}^{ev}_{hor}(P) \otimes \mathfrak{g})^G$ the curvature of θ.

The Chern-Weil morphism cw : $S(\mathfrak{g})^{Ad(G)} \to \mathcal{A}(P)^{ev}_{hor} \cong \mathcal{A}^{ev}(M)$ is given by

$$\mathrm{cw}(f)(\Theta) = f(\Theta_1, \cdots, \Theta_r).$$

This can be extended to $C^\infty(\mathfrak{g})^{Ad(G)}$ using Taylor expansions at 0.

Similarly defined is the Chern-Weil isomorphism

$$CW : \mathcal{H}^\infty_{G,c}(\mathfrak{g}, Q) \xrightarrow{\cong} \mathcal{H}_{dR,c}(Q/G),$$

for any G-principal bundle $Q \to Q/G$.

Twisted Chern character

- D family of $\tilde{G}$-transversally elliptic operators,
- $\mathrm{Ch}_\gamma^{\tilde{G}}(\sigma(D))(X) := \mathrm{Ch}_\gamma^{\tilde{G}}(\nabla^{\Pi^* E^+})(X) - \mathrm{Ch}_\gamma^{\tilde{G}}(\nabla^{\Pi^* E^-})(X) \in \mathcal{H}^\infty_{G,c}(\mathfrak{g}, T^*_G(P|B))$.
- $CW : \mathcal{H}^\infty_{G,c}(\mathfrak{g}, T^*_G(P|B)) \to \mathcal{H}_{dR,c}(T^*(M|B))$ the Chern-Weil isomorphism.

The following definition was introduced by Paradan in the case $B = *$.

Definition

The twisted Chern character

$$\mathrm{Ch}_\gamma(\sigma(D)) \in \mathcal{H}_{dR,c}(T^*(M|B))$$

is defined as the image of $\mathrm{Ch}_\gamma^{\tilde{G}}(\sigma(D)) \in \mathcal{H}^\infty_{G,c}(\mathfrak{g}, T^*_G(P|B))$ *under the Chern–Weil isomorphism.*

Distributional index of families of $\tilde{G}$-transversally elliptic operators

Let $D = \big(D_b : C^\infty(P_b, E_b^+) \to C^\infty(P_b, E_b^-)\big)_{b\in B}$ be a family of $\tilde{G}$-transversally elliptic operators. Then $\mathrm{Ind}_{-\infty}^{P|B}(D)$ is supported in Γ and therefore

$$\mathrm{Ind}_{-\infty}^{P|B}(D) = \sum_{\gamma\in\Gamma} Q_\gamma(D),$$

where $Q_\gamma(D) \in C^{-\infty}(\tilde{G})^{Ad(\tilde{G})} \otimes \mathcal{H}_{dR}(B,\mathbb{C})$ is supported in $\gamma \in \Gamma$.

Theorem (B. 2021)

For all $\varphi \in C^\infty(\tilde{G})^{Ad(\tilde{G})}$, we have

$$\langle Q_\gamma(D), \varphi\rangle = (2i\pi)^{-\dim(M|B)} \int_{T^*(M|B)|B} \mathrm{Ch}_\gamma(\sigma(D)) \wedge \mathrm{Td}(T^*(M|B)\otimes\mathbb{C}) \wedge \mathrm{cw}(\varphi(\gamma e^X)).$$

In particular, if $\varphi = 1$ in a neighborhood of γ then

$$\langle Q_\gamma(D), \varphi\rangle = (2i\pi)^{-\dim(M|B)} \int_{T^*(M|B)|B} \mathrm{Ch}_\gamma(\sigma(D)) \wedge \mathrm{Td}(T^*(M|B)\otimes\mathbb{C}).$$

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Family of $Spin(2n)$-transversally elliptic Dirac operators

- $1 \to \mathbb{Z}_2 \to Spin(2n) \xrightarrow{\zeta} SO(2n) \to 1$,
- $p : M \to B$ fibration of Riemannian, orientable, closed manifolds with $\dim(M|B) = 2n$,
- $\pi : \mathcal{F} := SO(M|B) \to M$ be the bundle of oriented orthonormal frame of $T(M|B)$.

Recall that any element of $\mathscr{E} \in \mathcal{F}$ is given by a linear isomorphism

$$\mathscr{E} : \mathbb{R}^{2n} \to T_{\pi(\mathscr{E})}(M|B),$$

and that

$$d_{\mathscr{E}}\pi : T_{SO(2n)}(\mathcal{F}|B)_{\mathscr{E}} \to T_{\pi(\mathscr{E})}(M|B)$$

is an isomorphism.
Therefore, we get a $SO(2n)$-equivariant trivialization

$$\alpha : T^*_{SO(2n)}(\mathcal{F}|B) \to \mathcal{F} \times (\mathbb{R}^{2n})^*,$$

given by $\alpha(\mathscr{E}, \xi) = (\mathscr{E}, \xi \circ (d_{\mathscr{E}}\pi)^{-1} \circ \mathscr{E})$.

Family of $Spin(2n)$-transversally elliptic Dirac operators

Let $S_{2n} = S_{2n}^+ \oplus S_{2n}^-$ be the spinor representation.
Denote by

$$cl : (\mathbb{R}^{2n})^* \longrightarrow \mathrm{End}(S_{2n}) \cong \mathbb{C}l(2n)$$

the Clifford action. Recall that cl is $Spin(2n)$-equivariant.
Then

$$\sigma(\not{\partial}^+_{\mathcal{F}|B})(\mathscr{E}, \xi) = cl(\xi \circ (d_{\mathscr{E}}\pi)^{-1} \circ \mathscr{E}) : S_{2n}^+ \to S_{2n}^-$$

defines the symbol of a family of $Spin(2n)$-transversally elliptic Dirac operators.

Theorem (B. 2021)

The distibutional index $\mathrm{Ind}_{-\infty}^{\mathcal{F}|B}(\not{\partial}^+_{\mathcal{F}|B}) = Q_{\mathrm{Id}} + Q_{-\mathrm{Id}}$ *of* $\not{\partial}^+_{\mathcal{F}|B}$ *is given by*

$$Q_{\pm\mathrm{Id}}(\varphi) = (2\pi i)^{-n} \int_{M|B} \hat{A}(T(M|B)) \wedge \mathrm{cw}(\varphi(\pm e^X)), \ \varphi \in C^\infty(Spin(2n))^{Ad(Spin(2n))}.$$

In particular, if $\varphi \in C^\infty(Spin(2n))^{Ad(Spin(2n))}$ *is a function equal to* 1 *around* Id *and* 0 *around* $-\mathrm{Id}$ *then*

$$\langle \mathrm{Ind}_{-\infty}^{\mathcal{F}|B}(\not{\partial}^+_{\mathcal{F}|B}), \varphi \rangle = (2\pi i)^{-n} \int_{M|B} \hat{A}(T(M|B)) \in \mathcal{H}^{ev}_{dR}(B).$$

Final comments

For any Azumaya bundle $\mathcal{A} \to M$, we can define a notion of families $A = (A_b)$ of projective operators which generalize the notion of projective operators introduced by Mathai-Melrose-Singer. We can define an index $\mathrm{Ind}_p^{M|B}(A) \in \mathcal{H}_{dR}(B)$ for such family.

To a family A of projective operators, we can associate a family $\tilde{A}$ of $SU(N)$-transversally elliptic operators on the $PU(N)$-principal bundle $\mathcal{P}$ associated with $\mathcal{A}$ and

$$\begin{aligned}\mathrm{Ind}_p^{M|B}(A) &= \langle \mathrm{Ind}_{-\infty}^{\mathcal{P}|B}(\tilde{A}), \varphi\rangle \\ &= (2i\pi)^{-\dim(M|B)} \int_{T^*(M|B)|B} \mathrm{Ch}_{\mathrm{Id}}(\sigma(\tilde{A})) \wedge \mathrm{Td}(T^*(M|B) \otimes \mathbb{C}).\end{aligned}$$

The Clifford bundle $\mathbb{C}l(M|B)$ is an Azumaya bundle and if M and B are oriented then M carries a family $\partial\!\!\!/_{M|B}$ of projective Dirac operators. Moreover,

$$\begin{aligned}\mathrm{Ind}_p^{M|B}(\partial\!\!\!/_{M|B}) &= \langle \mathrm{Ind}_{-\infty}^{\mathcal{F}|B}(\partial\!\!\!/^+_{\mathcal{F}|B}), \varphi\rangle \\ &= (2\pi i)^{-n} \int_{M|B} \hat{A}(T(M|B)) \in \mathcal{H}_{dR}^{ev}(B).\end{aligned}$$

Thank you for your attention.