Sobolev spaces and L^p-Poincaré inequalities in non-smooth spaces

Microlocal and global analysis, Applications to geometry

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Part 1: $(1, p)$ Sobolev regularity theory in MMS

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where λ_p is a (critical) exponent intrinsically associated with the small scale L^p-geometry of the underlying space.

The insight is that we replace the usual p-energy measure $\|\nabla f\|^p d\mu$ by a weak limit of the measures

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when $r \to 0$.

The Korevaar-Schoen energy approach to $(1, p)$ Sobolev regularity theory in MMS

For $\lambda > 0$ and $p \geq 1$, we define the space $KS^{\lambda,p}(X)$ as the collection of all functions $f \in L^p(X, \mu)$ for which

$$\limsup_{r \to 0^+} \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{r^\lambda \mu(B(x,r))} \, d\mu(y) \, d\mu(x) < +\infty.$$

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The L^p–Korevaar-Schoen critical exponent of the space (X, d, μ) is defined as

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The $(1,p)$ Korevaar-Schoen-Sobolev space is $W^{1,p}(X) := KS^{\lambda_p,p}(X)$.

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The case $p = 1$ is special and yields a theory of BV functions.

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Example 1: Let (X, d, μ) be a geodesically complete Riemannian manifold with non-negative Ricci curvature, equipped with its geodesic metric and Riemannian volume measure μ.

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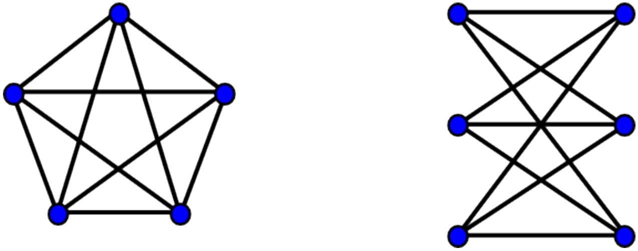

Figure 1: Cable system (picture from Wikipedia)

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&\simeq \sum_{i\in \mathbf{e}} \int_{\mathbf{e}_i} |f_i'(x)|^p dx
\end{aligned}$$

where f_i is the restriction of f to the edge $\mathbf{e}_i$.

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- The parameter $d_w \geq 2$ is called the walk dimension.

L^2 critical exponent in fractional spaces

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Moreover, on $W^{1,2}(X)$ one has the following equivalence of functionals,

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L^p critical exponents in strongly recurrent fractional spaces

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- If X is compact and there exists a non-constant $d_w - d_h$ Hölder continuous function in $W^{1,1}(X)$, then for every $p \geq 1$,

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- If for every $p \geq 1$, $\lambda_p = (p-1)d_w + (2-p)d_h$, then the family of Sobolev spaces $W^{1,p}(X)$ satisfy the Gagliardo-Nirenberg interpolation theory and associated Sobolev embeddings (B.-Alonso-Ruiz, 2020).

The Korevaar-Schoen energy approach to Sobolev regularity theory in fractals

We now look at a concrete example of strongly recurrent fractional metric spaces that come from the fractal world.

Vicsek set

Let $\{q_i\}_{1 \leq i \leq 5}$ be the 4 corners of a unit square and the center. Define

$$\psi_i(z) = \frac{1}{3}(z - q_i) + q_i, \quad 1 \leq i \leq 5.$$

The Vicsek set is the unique non-empty compact set such that

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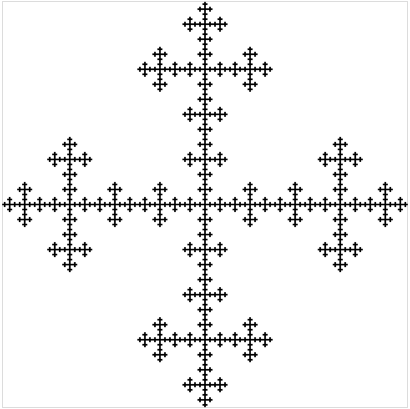

Figure 2: Vicsek set (picture from Wikipedia)

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- For $1 < p \leq 2$, the space $W^{1,p}(K)$ is dense in $L^p(K, \mu)$, see B.-Chen (2021).
- The case $p > 2$ was not explicitly discussed in the previous references (except as remarks), but similar arguments yield $\lambda_p = (p-1)d_w + (2-p)d_h$, $W^{1,p}(K) \subset C(K)$ and $W^{1,p}(K)$ is dense in $L^p(K, \mu)$.

Sierpiński gasket

Let $q_1 = 0, q_2 = 1, q_3 = e^{\frac{i\pi}{3}}$. Define

$$\psi_i(z) = \frac{1}{2}(z - q_i) + q_i, \quad i = 1, 2, 3.$$

The Sierpiński gasket is the unique non-empty compact set such that

$$K = \bigcup_{i=1}^{3} \psi_i(K).$$

Sierpiński gasket

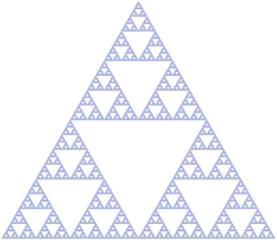

Figure 3: Sierpiński gasket (picture from Wikipedia)

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- It follows from Ju-Hi-Wen (2004), that for every $p > 1$, $W^{1,p}(K) \subset C(K)$ and that $W^{1,p}(K)$ is dense into $L^p(K, \mu)$.
- It follows from Herman-Peirone-Strichartz (2004) and ABCRST (2020), that for $p > 2$,

$$(p-1)d_w + (2-p)d_h < \lambda_p < \frac{1}{2} p d_w$$

and numerical simulations suggest that for $1 < p < 2$,

$$\frac{1}{2} p d_w < \lambda_p < (p-1)d_w + (2-p)d_h$$

Part 2: (p,p) Poincaré inequalities (based on a joint work with Li Chen, LSU)

L^p-Poincaré inequalities on balls

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$$\int_{B(x_0,R)} \left| f(x) - \fint_{B(x_0,R)} f d\mu \right|^p d\mu(x) \leq CR^{\lambda_p} \underline{\mathbf{Var}}_{B(x_0,AR),p}(f)^p,$$

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where the L^p Korevaar-Schoen (inner) variation of a Borel set F is defined by

$$\underline{\mathbf{Var}}_{F,p}(f)^p := \liminf_{r \to 0^+} \frac{1}{r^{\lambda_p}} \int_F \int_{B(x,r) \cap F} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} d\mu(x) d\mu(y).$$

A case of study: Simple nested fractals

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Self-similar structure: Consider similitudes $\{\psi_i\}_{i=1}^M$ with a common contraction factor $L > 1$. There is a unique compact set K such that

$$K = \bigcup_{i=1}^{M} \psi_i(K).$$

Unbounded nested fractals

By unbounded nested fractals we mean blow-ups of compact nested fractals. For instance, assume that $\psi_1 = L^{-1}x$ without loss of generality, then the unbounded nested fractal X will be defined by

$$X = \bigcup_{n=1}^{\infty} K^{\langle n \rangle} := \bigcup_{n=1}^{\infty} L^n K.$$

L^p-Poincaré inequalities

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As seen before, a concrete example would be the unbounded Vicsek set (or other treelike fractals).

Main theorem

Our main theorem is the following.

Theorem (B.-Chen 2020)

Let $1 < p \leq 2$. *There exist constants* $C > 0$ *and* $A > 1$ *such that for every* $f \in W^{1,p}(X)$, $x_0 \in X$, $R > 0$, $x, y \in B(x_0, R)$

$$|f(x) - f(y)| \leq CR^{(d_w - d_h)(1 - \frac{1}{p})} \underline{\mathbf{Var}}_{B(x_0, AR), p}(f).$$

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We note that the L^p-Poincaré inequality

$$\left\| f - \fint_{B(x_0,R)} f d\mu \right\|_{L^p(B(x_0,R),\mu)} \leq CR^{(d_w - d_h)\left(1-\frac{2}{p}\right)+\frac{d_w}{p}} \underline{\mathbf{Var}}_{B(x_0,AR),p}(f).$$

of course easily follows then from Hölder's inequality.

Application: Lusin-Hölder estimate

Theorem (B.-Chen 2020)

Let $1 < p \leq 2$. If $f \in W^{1,p}(X)$, there exists a function g which is in weak $L^p(X,\mu)$ such that for every $x, y \in X$

$$|f(x) - f(y)| \leq d(x,y)^{(d_w - d_h)\left(1-\frac{2}{p}\right)+\frac{d_w}{p}}(g(x)+g(y)).$$

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Theorem (B.-Chen 2020)

Let $1 < p \leq 2$. If $f \in W^{1,p}(X)$, there exists a function g which is in weak $L^p(X, \mu)$ such that for every $x, y \in X$

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For $p = 2$ and $d_w > 2$, the maximal function can not be in $L^2(K, \mu)$ unless f is constant.

Thank you !