Upper bound estimates of eigenvalues for Hörmander operators on non-equiregular sub-Riemannian manifolds

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Introduction

For $n \geq 2$, let M be a n-dimensional connected smooth manifold. We take a fixed smooth measure on M with strictly positive density, and write dx as the measure when we integrate. Also, we simply denote the measure of a set $E \subset M$ by $|E|$.

Let $X = (X_1, X_2, \cdots, X_m)$ be real C^∞ vector fields defined on M, which satisfy the following **Hörmander's condition** with Hörmander's index Q:

Hörmander's condition (Acta Math, 1967)

There exists a smallest positive integer Q such that these vector fields $X_1, X_2, \ldots, X_m$ together with their commutators of length at most Q span the tangent space $T_x(M)$ at each point of M.

Hörmander Operator $\triangle_X$

The formally self-adjoint Hörmander operator $\triangle_X$ generated by vector fields $X_1, X_2, \cdots, X_m$ is given by

$$\triangle_X := -\sum_{i=1}^{m} X_i^* X_i,$$

where X_i^* denotes the formal adjoint of X_i.

The study of Hörmander operators has received a strong impulse over the last fifty years. Motivation for the study of Hörmander operators among many different areas, e.g. **sub-Riemannian geometry, systems of stochastic differential equations, the theory of functions of several complex variables, geometric control theory and nonholonomic mechanics**. Through the work of several mathematicians much knowledge has been acquired, such as *subelliptic estimates, Poincaré inequalities, Sobolev embedding results, Harnack's inequalities, size estimates of the Green kernels and heat kernels*, etc.

sub-Riemannian Manifold

The vector fields $X = (X_1, X_2, \cdots, X_m)$ satisfying the Hörmander's condition on M induce a *sub-Riemannian manifold* (M, D, g) endowed with the *sub-Riemannian structure* (D, g), where the distribution D is a family of linear subspaces $D_x \subset T_x(M)$ such that

$$D_x = \text{span}\{X_1(x), X_2(x), \cdots, X_m(x)\}$$

depending smoothly on $x \in M$, and the sub-Riemannian metric $g : TM \to \mathbb{R} \cup \{+\infty\}$ is a function given by

$$g(x, v) = \inf \left\{ \sum_{i=1}^{m} u_i^2 \,\middle|\, v = \sum_{i=1}^{m} u_i X_i(x) \right\} \quad \text{for } x \in M, \ v \in T_x(M).$$

sub-Riemannian Manifold

We can see that $g(x,\cdot)$ is a positive definite quadratic form on D_x and $g(x,v)=+\infty$ for $v\notin D_x$. When D has a constant rank m on M with $m\le n$ (i.e. the dimension dim $D_x=m\le n$ for every $x\in M$), D is a subbundle of TM endowed with metric g. In this case, the Hörmander operator $\triangle_X=-\sum_{i=1}^{m}X_i^*X_i$ is the sub-Laplacian on sub-Riemannian manifold (M,D,g).

sub-Riemannian Distance

In the sub-Riemannian manifold (M, D, g), we can define the length of an absolutely continuous path $\varphi(t)$ $(0 \leq t \leq 1)$ in M by

$$\mathrm{length}(\varphi) = \int_0^1 \sqrt{g(\varphi(t), \varphi'(t))}dt.$$

Then, the sub-Riemannian distance d is given by

$$d(x, y) = \inf \mathrm{length}(\varphi), \tag{1.1}$$

where the infimum is taken on all the absolutely continuous paths joining x to y.

sub-Riemannian Distance

Consider an absolutely continuous path φ having finite length, we know $g(\varphi(t), \varphi'(t))$ is finite for almost any $t \in [0, 1]$. This implies $\varphi(t)$ satisfies the differential equation

$$\varphi'(t) = \sum_{i=1}^{m} a_i(t) X_i(\varphi(t)) \quad \text{a.e on } [0, 1], \tag{1.2}$$

with measurable control functions $a_1, \ldots, a_m \in L^1[0, 1]$. A path satisfying (1.1) is said to be *admissible*, or *controlled*.

sub-Riemannian Distance

The sub-Riemannian manifold (M, D, g) has a natural structure of metric space, where the distance is the so-called sub-Riemannian distance (or Carnot-Carathéodory distance).

Definition 1.1

Let $C_0(x, y)$ be the collection of absolutely continuous path $\varphi : [0, 1] \mapsto M$, which satisfies (1.2) with $\varphi(0) = x, \varphi(1) = y$ and m-tuples $(a_1, \ldots, a_m)$ of L^1 functions. Then the sub-Riemannian distance d can be given by

$$d(x, y) = \inf \left\{ \int_0^1 \sqrt{\sum_{i=1}^{m} a_i^2(t) dt} \middle| \varphi \in C_0(x, y) \right\}. \qquad (1.3)$$

sub-Riemannian Distance and Subunit Metric

Moreover, the vector fields X under the Hörmander's condition can induce another metric, namely the subunit metric.

Definition 1.2 (Subunit metric)

For any points $x, y \in M$ and $\delta > 0$, let $C(x, y, \delta)$ be the collection of absolutely continuous mapping $\varphi : [0, 1] \mapsto M$, which satisfies $\varphi(0) = x, \varphi(1) = y$ and

$$\varphi'(t) = \sum_{i=1}^{m} a_i(t) X_i(\varphi(t))$$

with $\sum_{k=1}^{m} |a_k(t)|^2 \leq \delta^2$ for a.e $t \in [0, 1]$. The subunit metric d_X is defined by

$$d_X(x, y) := \inf\{\delta > 0 \mid \exists \varphi \in C(x, y, \delta) \text{ with } \varphi(0) = x,\ \varphi(1) = y\}.$$

According to the result of Jerison[1], we know the Carnot-Carathéodory distance d is equivalent to the subunit metric d_X. Therefore, we can only pay attention to the subunit metric d_X.

[1] Subelliptic second order differential operators. In *Complex analysis III* (pp. 46-77). Springer, Berlin, Heidelberg

Subunit ball

Let

$$B_{d_X}(x,r) := \{y \in M \mid d_X(x,y) < r\}$$

be the subunit ball induced by the subunit metric $d_X(x,y)$. The volume of subunit ball appears in many estimates related to the sub-elliptic operator $\triangle_X$, such as the Gaussian bounds of sub-elliptic heat kernel, size estimates of Green function, bounds of counting functions (see Fefferman and Phong's result below), etc.

sub-Riemannian Manifold

For any $x \in M$, the **sub-Riemannian flag** at x is the sequence of nested vector subspaces

$$\{0\} = D_x^0 \subset D_x = D_x^1 \subset D_x^2 \subset \cdots \subset D_x^{r(x)-1} \subsetneq D_x^{r(x)} \subseteq T_x(M)$$

defined in terms of successive Lie brackets, and $r(x) \leq Q$ is the *degree of nonholonomy* at x. Here for each $1 \leq j \leq r(x)$, D_x^j is the subspaces of the tangent space at x spanned by all commutators of $X_1, \ldots, X_m$ with length at most j. Setting $\nu_j(x) = \dim D_x^j$ for $1 \leq j \leq r(x)$ with $\nu_0(x) := 0$, the integer

$$\nu(x) = \sum_{j=1}^{r(x)} j(\nu_j(x) - \nu_{j-1}(x)) \tag{1.4}$$

is called the **pointwise homogeneous dimension** at x.

sub-Riemannian Manifold

A point $x \in M$ is **regular** if, for every $1 \leq j \leq r(x)$, the dimension $\nu_j(y)$ is a constant as y varies in a open neighborhood of x. Otherwise, x is said to be **singular**. A set $S \subset M$ is **equiregular** if every point of S is regular. The set $S \subset M$ is said to be **non-equiregular** if it contains some singular points.

The equiregular assumption in sub-Riemannian geometry is also known as the **Métivier's condition** in PDEs. For the equiregular connected set S, we know that the pointwise homogeneous dimension $\nu(x)$ is a constant ν which coincides with the Hausdorff dimension of S related to the vector fields X, and this constant ν is also called the **Métivier's index**.

sub-Riemannian Manifold

Furthermore, if the set $S \subset M$ is non-equiregular, we can introduce the so-called **generalized Métivier's index** by

$$\tilde{\nu}_S := \max_{x \in \overline{S}} \nu(x). \tag{1.5}$$

The generalized Métivier's index is also known as the **non-isotropic dimension**, which plays a crucial role in the geometry and functional settings associated to the vector fields X. Note that

$$n + \max_{x \in \overline{S}} r(x) - 1 \le \tilde{\nu}_S < nQ$$

for $Q > 1$, and $\tilde{\nu}_S = \nu$ if the closure of S is equiregular and connected.

Eigenvalue problems of Hörmander operator $\triangle_X$

This work is concerned with the eigenvalue problems of $-\triangle_X$ on **non-equiregular** sub-Riemannian manifolds.

We first study the closed eigenvalue problem of $-\triangle_X$ on the compact manifold M without boundary, i.e.

Closed eigenvalue problem of $-\triangle_X$

$$-\triangle_X u = \mu u \qquad \text{in } M, \tag{1.6}$$

Eigenvalue problems of Hörmander operator $\triangle_X$

Secondly, we study the Dirichlet eigenvalue problem of $-\triangle_X$ on the compact manifold with boundary. For simplicity we assume that M is an open connected domain in $\mathbb{R}^n$ endowed with Lebesgue measure, and $\Omega \subset\subset M$ is a bounded connected open subset with smooth boundary such that $\partial\Omega$ is non-characteristic for $X = (X_1, X_2, \cdots, X_m)$. The Dirichlet eigenvalue problem of $-\triangle_X$ will be considered as follows:

Dirichlet eigenvalue problem of $-\triangle_X$

$$\begin{cases} -\triangle_X u = \lambda u, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Discreteness of Eigenvalues

We mention that, in both cases above, the Hörmander's condition ensures that the positive self-adjoint operator $-\triangle_X$ possesses discrete eigenvalues, and thus the eigenvalue problems (1.6) and (1.7) are well-defined.

Throughout this work, we let μ_k and λ_k be the k^{th} eigenvalue of the problems (1.6) and (1.7) respectively. Then we have

$$\begin{aligned} 0 &= \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k-1} \leq \mu_k \leq \cdots, \\ 0 &< \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{k-1} \leq \lambda_k \leq \cdots, \end{aligned}$$

and $\mu_k \to +\infty$, $\lambda_k \to +\infty$ as $k \to +\infty$.

Results for Classical Case $\triangle_X = \triangle$

If $X = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n})$, then $\triangle_X = \triangle$ is the classical Laplacian. In this classical case, H. Weyl gave the following asymptotic results.

Weyl's Law (Math. Ann. 1912)

- $$\mu_k \sim 4\pi^2 (|B_1||M|)^{-\frac{2}{n}} \cdot k^{\frac{2}{n}} \quad \text{as} \quad k \to +\infty, \tag{1.8}$$
- $$\lambda_k \sim 4\pi^2 (|B_1||\Omega|)^{-\frac{2}{n}} \cdot k^{\frac{2}{n}} \quad \text{as} \quad k \to +\infty. \tag{1.9}$$

where μ_k and λ_k are the k-th closed eigenvalue and Dirichlet eigenvalue of $-\triangle$, respectively. $|B_1|$ being the volume of the unit ball in $\mathbb{R}^n$.

Results for Classical Case $\triangle_X = \triangle$

The bound estimates of the eigenvalues have attracted a lot of attention since Pólya made his famous conjecture in 1961, namely

Pólya's conjecture (Proc. London Math. Soc. 1961)

$$\lambda_k \geq C_n \left(\frac{k}{|\Omega|}\right)^{\frac{2}{n}} \geq \mu_k, \text{ for } k \geq 1, \tag{1.10}$$

where $C_n = (2\pi)^2|B_1|^{-2/n}$, λ_k is the k-th Dirichlet eigenvalue and μ_k is the k-th Neumann eigenvalue.

Pólya proved (1.10) when Ω is assumed to be a plane domain which tiles $\mathbb{R}^2$ (and his proof also works in $\mathbb{R}^n$).

Results for Classical Case $\triangle_X = \triangle$

Lower bounds of eigenvalues for $\triangle$

- Cheng and Li[a] gave

$$\mu_k \geq C \cdot k^{\frac{2}{n}}, \qquad \forall k \geq 1, \tag{1.11}$$

where C is a positive constants depend on the constant in Sobolev inequality and M.

- Li-Yau inequality: By using the Fourier transformation approach, Li and Yau[b] gave a simple proof for the following lower bound

$$\sum_{i=1}^{k} \lambda_i \geq C_n \frac{n}{n+2} |\Omega|^{-\frac{2}{n}} k^{\frac{n+2}{n}}, \text{for any } k \geq 1. \tag{1.12}$$

[a] Comment. Math, 1981
[b] Comm. Math. Phys. 1983

Results for Classical Case $\triangle_X = \triangle$

Upper bounds of closed eigenvalues for $\triangle$

- For the classical closed eigenvalue problem of $\triangle$ on compact manifold M, Kröger[a] proved that

$$\sum_{j=1}^{k} \mu_j^2 \leq (2\pi)^2 \cdot \frac{n}{n+4} |M|^{-\frac{2}{n}} |B_1|^{-\frac{2}{n}} \cdot (k+1)^{1+\frac{4}{n}} \times \left(1 + C_0 \cdot \frac{(k+1)^{-\frac{1}{n}}}{|M|}\right) \tag{1.13}$$

holds for $k \geq 1$, where C_0 is a positive constant depending on compact manifold M.

[a] J. Funct. Anal. 1994

Results for Classical Case $\triangle_X = \triangle$

Upper bounds of Dirichlet eigenvalues for $\triangle$

- For the classical Dirichlet eigenvalue problem of $\triangle$ on Ω, under some assumptions, Kröger[a] also obtained that

$$\sum_{j=1}^{k} \lambda_j \le \frac{n}{n+2}(2\pi)^2 |B_1|^{-\frac{2}{n}} \cdot k^{1+\frac{2}{n}} \cdot |\Omega|^{-\frac{2}{n}} + \widetilde{C_0} k^{1+\frac{1}{n}} \tag{1.14}$$

holds for sufficient large k, where $\widetilde{C_0}$ is a positive constants depending on Ω.

- Laptev[b] proved

$$k \ge (\lambda_k - \lambda_1)^{\frac{n}{2}} |B_1| (2\pi)^{-n} \frac{2}{n+2} \cdot \left(\sup_{x \in \Omega} |\phi_1(x)| \right)^{-2}. \tag{1.15}$$

Here ϕ_1 is the first Dirichlet eigenfunction corresponds with λ_1.

[a] J. Funct. Anal. 1994
[b] J. Funct. Anal. 1997

Results for Classical Case $\triangle_X = \triangle$

Upper bounds for the Dirichlet eigenvalues of $\triangle$

- Hermi[a] obtained

$$\frac{\lambda_{k+1}}{\lambda_1} \leq \left(1+\frac{n}{2}\right)^{\frac{2}{n}} \left(\frac{2n}{j^2_{\frac{n}{2}-1,1} J^2_{\frac{n}{2}}(j_{\frac{n}{2}-1,1})}\right)^{\frac{2}{n}} \cdot k^{\frac{2}{n}} + 1 \quad \text{for all } k \geq 1, \tag{1.16}$$

where $j_{p,k}$ denotes the k-th positive zero of the standard Bessel function $J_p(x)$ of the first kind of order p.

[a] Trans. Amer. Math. Soc. (2008)

Results for Classical Case $\triangle_X = \triangle$

Upper bounds for the Dirichlet eigenvalues of $\triangle$

- Cheng and Yang[a] proved that

$$\frac{\lambda_{k+1}}{\lambda_1} \le C_0(n,k)k^{\frac{2}{n}}, \quad \text{for all } k \ge 1, \tag{1.17}$$

where the constant

$$C_0(n,k) = \begin{cases} \frac{j^2_{\frac{n}{2},1}}{j^2_{\frac{n}{2}-1,1}}, & \text{for } k=1; \\ 1 + \frac{a(\min\{n,k-1\})}{n}, & \text{for } k \ge 2. \end{cases}$$

and $a(1) \le 2.64$, $a(2) \le 2.27$ and $a(p) \le 2.2 - 4\log\left(1 + \frac{p-3}{50}\right)$ for $p \ge 3$ is a constant depending only on p.

[a] Math. Ann. (2007)

Results for Finite Degenerate Case

If the sub-Riemannian manifold M is assumed to be **equiregular** (i.e. the vector fields X satisfy the Métivier's condition on M), G. Métivier[2] proved the following asymptotic results.

Métivier's asymptotic results

Denote by μ_k and λ_k the k-th closed eigenvalue and k-th Dirichlet eigenvalue of $-\triangle_X$, respectively. Then we have

$$\mu_k \sim \left(\int_\Omega \gamma(x)dx\right)^{-\frac{2}{\nu}} \cdot k^{\frac{2}{\nu}}, \text{ as } k \to +\infty, \tag{1.18}$$

and

$$\lambda_k \sim \left(\int_\Omega \gamma(x)dx\right)^{-\frac{2}{\nu}} \cdot k^{\frac{2}{\nu}}, \text{ as } k \to +\infty, \tag{1.19}$$

where $\gamma(x)$ is a positive continuous function on M (resp. Ω), ν is the Métivier's index which is also called the **Hausdorff dimension** (or **homogeneous dimension**) of Ω related to the subelliptic metric induced by the vector fields X.

[2] Comm. Partial Differential Equations, 1976

On Equiregular Assumption (Métivier's condition)

We must point out that the equiregular assumption (Métivier's condition) is a strong assumption in sub-Riemannian manifold (M, D, g), under this assumption the Lie algebra generated by the vector fields $X_1, X_2, \dots, X_m$ has constant structure, and the vector fields can be well locally approximated by some homogeneous left invariant vector fields defined on corresponding Carnot group (See Rothschild and Stein, Acta Math. 1976). However there also exist many vector fields which only satisfy the Hörmander's condition but not satisfy the Métivier's condition (e.g. the Grushin type vector fields $X_1 = \partial_{x_1}, X_2 = x_1\partial_{x_2}$ in $\mathbb{R}^2$). In this case, the corresponding sub-Riemannian manifold (M, D, g) is non-equiregular.

Results for Finite Degenerate Case

For the Hörmander operator $\triangle_X$ on general **non-equiregular** sub-Riemannian manifold M, Fefferman and Phong[3] studied the closed eigenvalue problem (1.6) and gave

Fefferman and Phong's abstract estimate

$$c_1 \int_M \frac{dx}{|B_{d_X}(x, \lambda^{-\frac{1}{2}})|} \leq N(\lambda) \leq c_2 \int_M \frac{dx}{|B_{d_X}(x, \lambda^{-\frac{1}{2}})|} \tag{1.20}$$

holds for sufficient large λ, where $N(\lambda) := \#\{k|\mu_k \leq \lambda\}$ is the counting function, $c_2 > c_1 > 0$ are some constant depending on X and M, and $B_{d_X}(x, r)$ is the subunit ball.

[3] Conference on harmonic analysis in honor of Antoni Zygmund. 1981

Results for Finite Degenerate Case

Recently, we obtain the following results for the closed eigenvalue problem (1.6).

Proposition 1.1 (Chen-Chen, J .Math.Pures Appl. 2019)

Let $X = (X_1, X_2, \cdots, X_m)$ satisfy the Hörmander's condition on compact manifold M with Hörmander index $Q \geq 2$. Denote by μ_k the k-th closed eigenvalue of $-\triangle_X$ on M, then for any $k \geq 1$ we have

$$\sum_{i=1}^{k} \mu_i \geq C_1 \cdot k^{1+\frac{2}{\tilde{\nu}}}, \tag{1.21}$$

where C_1 is a positive constant depending on M and X, $\tilde{\nu}$ is the non-isotropic dimension of M related to the vector fields X.

Remark 1.1

Since $k\mu_k \geq \sum_{j=1}^{k} \mu_j$, we have $\mu_k \geq Ck^{\frac{2}{\tilde{\nu}}}$ for all $k \geq 1$.

Results for Finite Degenerate Case

Proposition 1.2 (Chen-Chen, J .Math.Pures Appl. 2019)

Let $X = (X_1, X_2, \cdots, X_m)$ *satisfy the Hörmander's condition on compact manifold* M *with Hörmander index* $Q \geq 2$. *Then for the closed eigenvalue problem* (1.6), *there exists a positive measurable function* $c_0(x)$ *on* M *such that*

$$\lim_{\lambda \to +\infty} \lambda^{-\frac{\tilde{\nu}}{2}} N(\lambda, -\triangle_X) = \frac{\int_H c_0(x)dx}{\Gamma\left(\frac{\tilde{\nu}}{2} + 1\right)}. \tag{1.22}$$

Here $\tilde{\nu} := \max_{x \in M} \nu(x)$ *is the non-isotropic dimension of* M, $H = \{x \in M \mid \nu(x) = \tilde{\nu}\} \subset M$ *is the level set of maximum value of pointwise homogeneous dimension, and* $N(\lambda, -\triangle_X) = \#\{k \mid 0 \leq \mu_k \leq \lambda\}$.

Results for Finite Degenerate Case

Furthermore, we can deduce from Proposition 1.2 that

Asymptotic of Subelliptic Closed Eigenvalues

- If $|H| > 0$, we have

$$\mu_k = \left(\frac{\Gamma\left(\frac{\tilde{\nu}}{2}+1\right)}{\int_H c_0(x)dx} \right)^{\frac{2}{\tilde{\nu}}} \cdot k^{\frac{2}{\tilde{\nu}}} + o(k^{\frac{2}{\tilde{\nu}}}), \quad \text{as } k \to +\infty; \tag{1.23}$$

- If $|H| = 0$, then we have

$$\lim_{k\to+\infty} \frac{k^{\frac{2}{\tilde{\nu}}}}{\mu_k} = 0. \tag{1.24}$$

Results for Finite Degenerate Case

For the sub-elliptic Dirichlet eigenvalue problem (1.7), we also have the following sharp lower bound.

Proposition 1.3 (Chen-Chen, Proc. Lond. Math. Soc. 2021)

Let $X = (X_1, X_2, \cdots, X_m)$ *satisfy the Hörmander's condition in an open domain* $M \subset \mathbb{R}^n$ *with Hörmander's index* $Q \geq 2$, $\Omega \subset\subset M$ *be a connected bounded open subset with* C^∞ *non-characteristic boundary for* X. *Denote by* λ_k *the* k^{th} *Dirichlet eigenvalue of problem* (1.7) *on* Ω. *Then we have*

$$\sum_{j=1}^{k} \lambda_j \geq C \cdot k^{1+\frac{2}{\tilde{\nu}}}, \qquad \forall\ k \geq 1, \tag{1.25}$$

where $C = C(X, \Omega) > 0$, $\tilde{\nu}$ *is the non-isotropic dimension of* Ω *related to the vector fields* X.

Remark 1.2

Since $k\lambda_k \geq \sum_{j=1}^{k} \lambda_j$, *we also have* $\lambda_k \geq Ck^{\frac{2}{\tilde{\nu}}}$ *for all* $k \geq 1$.

Results for Finite Degenerate Case

Similarly, we have the following asymptotic results for the sub-elliptic Dirichlet eigenvalue problem (1.7).

Proposition 1.4 (Chen-Chen, Proc. Lond. Math. Soc. 2021)

Suppose that $X = (X_1, X_2, \cdots, X_m)$ *and* Ω *satisfy the conditions in Proposition 1.3. Then there exists a positive measurable function* γ_0 *for all* $x \in \Omega$, *such that*

$$\lim_{\lambda \to +\infty} \lambda^{-\frac{\tilde{\nu}}{2}} N(\lambda) = \frac{1}{\Gamma\left(\frac{\tilde{\nu}}{2}+1\right)} \cdot \int_H \gamma_0(x)dx. \qquad (1.26)$$

Here $H = \{x \in \Omega \mid \nu(x) = \tilde{\nu}\} \subset \Omega$ *is the level set of maximum value of pointwise homogeneous dimension, and* $N(\lambda) = \#\{k \mid 0 < \lambda_k \le \lambda\}$.

Results for Finite Degenerate Case

Moreover, we can deduce from Proposition 1.4 that

Asymptotic of Subelliptic Dirichelt Eigenvalues

- If $|H| > 0$, we have

$$\lambda_k = \left(\frac{\Gamma\left(\frac{\tilde{\nu}}{2} + 1\right)}{\int_H \gamma_0(x)dx} \right)^{\frac{2}{\tilde{\nu}}} \cdot k^{\frac{2}{\tilde{\nu}}} + o(k^{\frac{2}{\tilde{\nu}}}), \quad \text{as } k \to +\infty; \tag{1.27}$$

- If $|H| = 0$, then we have

$$\lim_{k \to +\infty} \frac{k^{\frac{2}{\tilde{\nu}}}}{\lambda_k} = 0. \tag{1.28}$$

Some Remarks

Remark

From the results above, we know that H would be the most singular part of M. If H has positive measure, the lower bounds (1.21) and (1.25) for μ_k and λ_k are optimal in the sense of the order of k. In particular, when the manifold M is **equiregular**, we know that $H = M$ (resp. $H = \Omega$) and our asymptotic formulas (1.23) and (1.27) are coincide with the Métivier's asymptotic estimates (1.18) and (1.19). If H has zero measure, then the results above imply that our lower bounds (1.21) and (1.25) for μ_k and λ_k are not precise since $\mu_k^{-1} = o(k^{-\frac{2}{\tilde{\nu}}})$ (resp. $\lambda_k^{-1} = o(k^{-\frac{2}{\tilde{\nu}}})$) as $k \to +\infty$.

Some Remarks

Corollary 1.1

For the closed eigenvalues μ_k (resp. Dirichlet eigenvalues λ_k) of subelliptic operator $-\triangle_X$ on M (resp. Ω), the result $\mu_k \approx k^{\frac{2}{\tilde{\nu}}}$ (resp. $\lambda_k \approx k^{\frac{2}{\tilde{\nu}}}$) holds as $k \to +\infty$ if and only if $|H| > 0$.

Remark 1.3

The result of Corollary 1.1 has the following geometric meaning: Under the condition $|H| > 0$, the non-isotropic dimension $\tilde{\nu}$ of M (resp. Ω) related to X will be a spectral invariant.

Results for Finite Degenerate Case

By the following approaches of Korevaar[4] and Grigor'yan-Yau [5], Kokarev [6] obtained the upper bounds of Neumann eigenvalues for the sub-Laplacian on compact strictly pseudo-convex CR manifold of dimension $2n+1$ with boundary.

Upper bounds of Neumann eigenvalues on CR manifold

Denote by μ_k the k-th Neumann eigenvalue of sub-Laplacian $-\triangle_b$ on compact **strictly pseudo-convex CR manifold** of dimension $2n+1$ with boundary (with $\nu = 2n+2$, $\frac{2}{\nu} = \frac{1}{n+1}$). Then we have

$$\mu_k |M|^{\frac{1}{n+1}} \leq C_* \cdot k^{\frac{1}{n+1}}, \quad \forall k \geq 1, \tag{1.29}$$

where C_* is a positive constant depending on CR structure.

Later, Kokarev's result was generalized by Hassannezhad and Kokarev [7] to the sub-Laplacians on **equiregular** sub-Riemannian manifolds.

[4] J. Differential Geom. (1993)
[5] Surveys in Differential Geometry, Vol. IX.
[6] Comm. Partial Differential Equations. (2013)
[7] Ann. Sc. Norm. Super. Pisa Cl. Sci. (2014)

Main Results

Recently, we obtain the following estimate for closed eigenvalue problem (1.6).

Theorem 1 (Chen-Chen-Li, 2021)

Let $X = (X_1, X_2, \cdots, X_m)$ *be the* C^∞ *real vector fields defined on the compact manifold* M*, which satisfy the Hörmander's condition in* M*. Then for any* $0 < t < 1$ *and any* $k \geq 0$*, we have*

$$\mu_{k+1}\left[A_1\int_M \frac{dx}{|B_{d_X}(x,\sqrt{t})|} - (k+1)\right] + \sum_{j=0}^{k}\mu_j \leq \frac{C_{1,1}}{t}\int_M \frac{dx}{|B_{d_X}(x,\sqrt{t})|}, \tag{2.1}$$

where A_1 *and* $C_{1,1}$ *are some positive constants depending only on the sub-Riemannian structure.* $B_{d_X}(x,r)$ *is the subunit ball induced by the subunit metric* d_X*.*

This is extension of Fefferman and Phong's result (1.20).

Main Results

Remark 2.1

Theorem 1 gives an estimate of eigenvalues which depends on the volume of subunit ball. From Theorem 1, we can actually recover the lower bound estimate of the counting function $N(\lambda)$ given by Fefferman and Phong in (1.20)*. This means* (2.1) *possesses the optimal growth order.*

Then, if the level set of maximum value of pointwise homogeneous dimension H has a positive measure, Theorem 1 yields the following explicit upper bounds of eigenvalues for the sub-elliptic closed eigenvalue problem (1.6).

Theorem 2 (Chen-Chen-Li, 2021)

Suppose that $X = (X_1, X_2, \cdots, X_m)$ and M satisfy the conditions in Theorem 1. Denote by $\tilde{\nu} = \max_{x \in M} \nu(x)$ the generalized Métivier's index of M. If the set $H := \{x \in M | \nu(x) = \tilde{\nu} = \max_{x \in M} \nu(x)\}$ possesses a positive measure, then for any $k \geq 1$ we have

$$\sum_{j=1}^{k} \mu_j \leq \frac{C_{1,1}}{\widehat{C_1}} \cdot \left(\frac{\widehat{C_2}}{A_1} \right)^{1+\frac{2}{\tilde{\nu}}} \cdot \frac{|M|}{|H|^{1+\frac{2}{\tilde{\nu}}}} \cdot (k+1)^{1+\frac{2}{\tilde{\nu}}} \tag{2.2}$$

and

$$\mu_k \leq \frac{C_{1,1}}{\widehat{C_1}} \cdot \left(\frac{2\widehat{C_2}}{A_1} \right)^{1+\frac{2}{\tilde{\nu}}} \cdot \frac{|M|}{|H|^{1+\frac{2}{\tilde{\nu}}}} \cdot k^{\frac{2}{\tilde{\nu}}}, \tag{2.3}$$

where $\widehat{C_1}$ and $\widehat{C_2}$ are some positive constants depending only on the sub-Riemannian structure.

Remark 2.2

By the asymptotic formula in (1.23) above, we see that the upper bounds (2.2) and (2.3) for μ_k in Theorem 2 are optimal in terms of the order on k. In particular, if M is equiregular, then $\tilde{\nu} = \nu$ and $H = M$. In this case, the upper bound estimate (2.3) above gives the similar result with the main results of Hassannezhad and Kokarev [a][b]. Nevertheless, our results in this paper will be suitable for general equiregular sub-Riemannian manifolds and non-equiregular sub-Riemannian manifolds with $|H| > 0$.

[a] Ann. Sc. Norm. Super. Pisa Cl. Sci. (2014)
[b] Comm. Partial Differential Equations. (2013)

On the other hand, we obtain the following inequality for the Dirichlet eigenvalues of the problem (1.7).

Theorem 3 (Chen-Chen-Li, 2021)

Let $X = (X_1, X_2, \cdots, X_m)$ be C^∞ real vector fields defined on a connected open domain M in $\mathbb{R}^n$, which satisfy the Hörmander's condition. Assume that $\Omega \subset\subset M$ is a bounded connected open subset with smooth boundary such that $\partial\Omega$ is non-characteristic for X. Then for any compact subset $K \subset \Omega$, there exists a positive constant $\delta(K)$, such that for any $0 < t \leq \delta(K)$ and any $k \geq 1$, we have

$$\lambda_{k+1}\left[\frac{A_1}{2}\cdot\int_K \frac{dx}{|B_{d_X}(x,\sqrt{t})|} - k\right] + \sum_{j=1}^{k}\lambda_j \leq \frac{2A_3}{t}\int_\Omega \frac{1}{|B_{d_X}(x,\sqrt{t})|}dx, \tag{2.4}$$

where A_1 and A_3 are some positive constants depending only on the sub-Riemannian structure.

Similarly, when the set H possesses a positive measure, Theorem 3 also indicates the following explicit upper bounds which are compatible with the asymptotic formula in (1.27).

Theorem 4 (Chen-Chen-Li, 2021)

Suppose that $X = (X_1, X_2, \cdots, X_m)$ and Ω satisfy the conditions in Theorem 3. Let $\tilde{\nu} = \max_{x\in\overline{\Omega}} \nu(x)$ be the non-isotropic dimension of Ω related to the vector fields X. If the set $H := \{x \in \Omega | \nu(x) = \tilde{\nu} = \max_{x\in\overline{\Omega}} \nu(x)\}$ admits a positive measure, then for any $k \geq 1$ we have

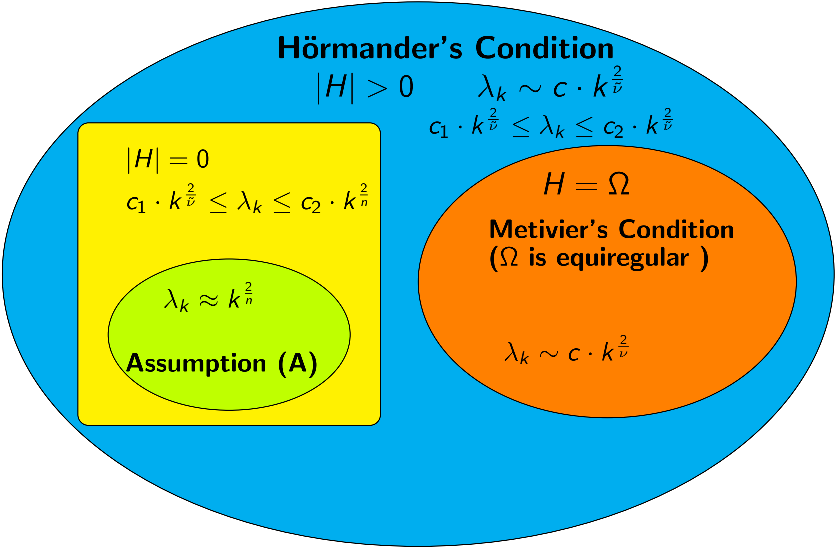
$$\sum_{j=1}^{k} \lambda_j \leq \frac{2A_3}{\widehat{C_1}} \cdot \left(\frac{4\widehat{C_2}}{A_1}\right)^{1+\frac{2}{\tilde{\nu}}} \cdot \frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}} \cdot k^{1+\frac{2}{\tilde{\nu}}} \tag{2.5}$$

and

$$\lambda_{k+1} \leq \frac{2A_3}{\widehat{C_1}} \cdot \left(\frac{8\widehat{C_2}}{A_1}\right)^{1+\frac{2}{\tilde{\nu}}} \cdot \frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}} \cdot k^{\frac{2}{\tilde{\nu}}}, \tag{2.6}$$

where $\widehat{C_1}$ and $\widehat{C_2}$ are some positive constants depending only on the sub-Riemannian structure.

Relations of Results for Sub-elliptic Operators



Some Examples

Example 3.1 (Math. Ann. (1989) 285(5), 447-461)

Let $\Omega \subset \mathbb{R}^3$ *be a bounded connected open set with a smooth boundary* $\partial\Omega$ *that is non-characteristic for* X. *Moreover, assume that* $B_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sqrt{x_1^2 + x_2^2 + x_3^2} < 2\} \subset\subset \Omega$. *Give the vector fields* $X = \{X_1, X_2, X_3\}$ *defined in* $\mathbb{R}^3$ *such that*

$$X_1 = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1\frac{\partial}{\partial x_3},$$

$$X_3 = (\phi_1(x_1, x_2) + \phi_2(x_3) + \phi_3(x_3))\frac{\partial}{\partial x_3},$$

where

$$\phi_1(x_1, x_2) = \begin{cases} e^{-(\ln(\sqrt{x_1^2+x_2^2}-\frac{3}{2}))^2}, & \sqrt{x_1^2 + x_2^2} > \frac{3}{2}; \\ 0, & \sqrt{x_1^2 + x_2^2} \le \frac{3}{2}. \end{cases}$$

$$\phi_2(x_3) = \begin{cases} e^{-(\ln x_3)^2}, & x_3 \in (0, +\infty); \\ 0, & x_3 \in (-\infty, 0]. \end{cases}$$

and

$$\phi_3(x_3) = \begin{cases} e^{-(\ln(-x_3-1))^2}, & x_3 \in (-\infty, -1); \\ 0, & x_3 \in [-1, +\infty). \end{cases}$$

From the assumptions above, the vector fields $X = (X_1, X_2, X_3)$ verify Hörmander's condition on $\overline{\Omega}$ with Hörmander index $Q = 2$. Setting

$$H := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \sqrt{x_1^2 + x_2^2} \leq \frac{3}{2}, -1 \leq x_3 \leq 0 \right\},$$

we have $H \subset\subset B_2 \subset\subset \Omega$ and

$$\nu(x) = \sum_{j=1}^{2} j(\nu_j(x) - \nu_{j-1}(x)) = \begin{cases} 4, & \text{if } x \in H \ ; \\ 3, & \text{if } x \in \overline{\Omega} \setminus H \ . \end{cases}$$

Thus, Ω is non-equiregular with respect to vector fields $X=(X_1,X_2,X_3)$. It follows that the generalized Métivier index $\tilde{\nu}=\max_{x\in\overline{\Omega}}\nu(x)=4$ and $|H|=\frac{9}{4}\pi>0$.

If we consider the Dirichlet eigenvalue problem (1.3) for the sub-elliptic operator $\triangle_X=-\sum_{j=1}^{3}X_j^{*}X_j$ on Ω, by Theorem 4 we obtain

$$\sum_{j=1}^{k}\lambda_j\le\frac{2A_3}{\widehat{C_1}}\cdot\left(\frac{4\widehat{C_2}}{A_1}\right)^{1+\frac{2}{\tilde{\nu}}}\cdot\frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}}\cdot k^{1+\frac{2}{\tilde{\nu}}}=\frac{2A_3}{\widehat{C_1}}\left(\frac{4\widehat{C_2}}{A_1}\right)^{\frac{3}{2}}|\Omega|\left(\frac{9\pi}{4}\right)^{-\frac{3}{2}}k^{\frac{3}{2}} \tag{3.1}$$

and

$$\lambda_{k+1}\le\frac{2A_3}{\widehat{C_1}}\cdot\left(\frac{8\widehat{C_2}}{A_1}\right)^{1+\frac{2}{\tilde{\nu}}}\cdot\frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}}\cdot k^{\frac{2}{\tilde{\nu}}}=\frac{2A_3}{\widehat{C_1}}\left(\frac{8\widehat{C_2}}{A_1}\right)^{\frac{3}{2}}|\Omega|\left(\frac{9\pi}{4}\right)^{-\frac{3}{2}}k^{\frac{1}{2}}, \tag{3.2}$$

where $A_1,A_3,\widehat{C_1},\widehat{C_2}$ are some positive constants depending on the sub-Riemannian structure induced by vector fields X on Ω.

Example 3.2

Let $X_1 = \partial_{x_1}, X_2 = \phi(x_1)\partial_{x_2}, X_3 = x_1\partial_{x_3} + (1-\phi(x_1))\partial_{x_2}$ *be the vector fields defined in* $\mathbb{R}^3$*, where*

$$\phi(x_1) := \begin{cases} e^{1-\frac{1}{1-x_1^2}}, & -1 < x_1 < 1; \\ 0, & others. \end{cases}$$

Since $\det(X_1, X_2, X_3)(x) = x_1\phi(x_1)$*, which gives*

$$\nu_1(x) = \begin{cases} 2, & x_1 = 0 \text{ or } |x_1| \geq 1; \\ 3, & 0 < |x_1| < 1. \end{cases}$$

Observing that $[X_1, X_2] = \phi'(x_1)\partial_{x_2}, [X_1, X_3] = \partial_{x_3} - \phi'(x_1)\partial_{x_2}, [X_2, X_3] = 0$*, we have*

$$\det(X_1, X_3, [X_1, X_3])(x) = 1 - \phi(x_1) + x_1\phi'(x_1) > 0 \quad for \quad |x_1| \geq 1,$$

$$\det(X_1, X_2, [X_1, X_3])(x) = \phi(x_1) > 0 \quad for \quad x_1 = 0.$$

That means $\nu_2(x) = 3$ for all $x \in \mathbb{R}^3$ and vector fields $X = (X_1, X_2, X_3)$ satisfy Hörmander's condition in $\mathbb{R}^3$ with Hörmander index $Q = 2$. Moreover, we have

$$\nu(x) = \sum_{j=0}^{2} j(\nu_j(x) - \nu_{j-1}(x)) = \begin{cases} 4, & x_1 = 0 \text{ or } |x_1| \geq 1; \\ 3, & 0 < |x_1| < 1. \end{cases}$$

Let Ω be a bounded open domain in $\mathbb{R}^3$ which contains $D(2) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | |x_j| < 2,\ j = 1, 2, 3\}$, we further assume that Ω has smooth boundary $\partial\Omega$ which is non-characteristic for $X = (X_1, X_2, X_3)$. Clearly, Ω is non-equiregular with respect to vector fields $X = (X_1, X_2, X_3)$, and the generalized Métivier index $\tilde{\nu} = \max_{x \in \overline{\Omega}} \nu(x) = 4$. Additionally, we have

$$H = \{x \in \Omega | \nu(x) = \tilde{\nu}\} = \{x \in \Omega | x_1 = 0 \text{ or } |x_1| \geq 1\},$$

which implies $H \neq \Omega$ but

$$|H| \geq |\{x \in D(2) | x_1 = 0 \text{ or } |x_1| \geq 1\}| = 32 > 0$$

.

If we consider the Dirichlet eigenvalue problem (1.3) for the sub-elliptic operator $\triangle_X = -\sum_{j=1}^{3} X_j^* X_j$ on Ω, Theorem 4 also gives

$$\sum_{j=1}^{k} \lambda_j \leq \frac{2A_3}{\widehat{C_1}} \cdot \left(\frac{4\widehat{C_2}}{A_1} \right)^{1+\frac{2}{\tilde{\nu}}} \cdot \frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}} \cdot k^{1+\frac{2}{\tilde{\nu}}} = \frac{2A_3}{\widehat{C_1}} \left(\frac{4\widehat{C_2}}{A_1} \right)^{\frac{3}{2}} |\Omega| \cdot |H|^{-\frac{3}{2}} k^{\frac{3}{2}} \tag{3.3}$$

and

$$\lambda_{k+1} \leq \frac{2A_3}{\widehat{C_1}} \cdot \left(\frac{8\widehat{C_2}}{A_1} \right)^{1+\frac{2}{\tilde{\nu}}} \cdot \frac{|\Omega|}{|H|^{1+\frac{2}{\tilde{\nu}}}} \cdot k^{\frac{2}{\tilde{\nu}}} = \frac{2A_3}{\widehat{C_1}} \left(\frac{8\widehat{C_2}}{A_1} \right)^{\frac{3}{2}} |\Omega| \cdot |H|^{-\frac{3}{2}} k^{\frac{1}{2}}, \tag{3.4}$$

where $A_1, A_3, \widehat{C_1}, \widehat{C_2}$ are some positive constants depending on the sub-Riemannian structure induced by vector fields X on Ω.

Example 3.3 (Grushin operator with $|H| = 0$)

For $n = 2$, let the Grushin vector fields $X = (\partial_{x_1}, x_1\partial_{x_2})$ defined in $\mathbb{R}^2$. The Grushin operator induced by X is defined by

$$\triangle_X := \frac{\partial^2}{\partial x_1^2} + x_1^2 \frac{\partial^2}{\partial x_2^2}. \tag{3.5}$$

Assume $\Omega \subset \mathbb{R}^2$ to be a bounded connected open subset with smooth boundary which is non-characteristic for X. We further assume that Ω contains the origin. One can easily verify that

$$\nu(x) = \begin{cases} 3, & \text{if } x_1 = 0; \\ 2, & \text{if } x_1 \neq 0, \end{cases}$$

which gives that $\tilde{\nu} = \max_{x \in \overline{\Omega}} \nu(x) = 3$ and $|H| = 0$.

From the result in our Proc. LMS paper, we have the following lower bound for Dirichlet eigenvalues:

$$\lambda_k \geq c_1 k \cdot (\ln k)^{-1}, \text{ for } k \geq 1, \text{ and some } c_1 > 0.$$

On the other hand, for this example of $|H| = 0$, we can prove the following upper bound for some positive constants a, b and C:

$$\lambda_{k+1} \leq C \cdot \frac{\ln\left(1 + a\sqrt{\frac{k+2}{\ln(k+2)}}\right)}{\ln\left(1 + b\sqrt{\frac{k+2}{\ln(k+2)}}\right)} \cdot \frac{k+2}{\ln(k+2)} \quad \text{for all} \quad k \geq 1,$$

which means $\lambda_k \leq c_2 k \cdot (\ln k)^{-1}$ for some positive constant $c_2 > 0$. That means $\lambda_k \approx k \cdot (\ln k)^{-1}$ in this case.

Thank you for your attention!