# Mode stability and shallow quasinormal modes of Kerr–de Sitter black holes

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February 23, 2022

# Introduction and motivation

Goal: study long time asymptotics of linear waves propagating on certain black hole spacetimes of General Relativity (GR).

1. Do waves decay at all? ('Linear stability.') Highly nontrivial in the absence of (coercive) conserved energies.
2. If yes, what is the decay rate? Can one prove asymptotic expansions?

Typical applications:

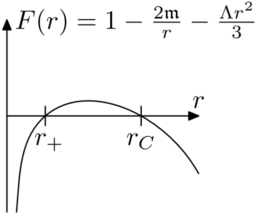
1. global existence results for nonlinear wave equations;
2. in the context of Einstein's field equations: nonlinear stability of the (family of) spacetime(s) under consideration.

# Black holes in de Sitter space

Fix the cosmological constant $\Lambda > 0$.

Schwarzschild–de Sitter (SdS). Black hole mass $\mathfrak{m} \in (0, (9\Lambda)^{-1/2})$.

1. Metric: $g = -F(r)\,\mathrm{d}t^2 + F(r)^{-1}\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}$, where



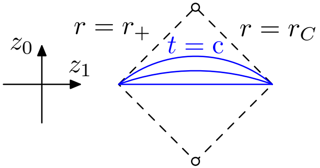

2. Manifold: $\mathbb{R}_t \times (r_+, r_C)_r \times \mathbb{S}^2$. Have $r_+ \simeq 2\mathfrak{m}$, $r_C \simeq \sqrt{\frac{3}{\Lambda}}$.
3. Metric is stationary, spherically symmetric. $\mathrm{Ric}(g) - \Lambda g = 0$.

Kerr–de Sitter (KdS). Angular momentum $|\mathfrak{a}| \lesssim \mathfrak{m}$. Explicit metric (Carter '68), same manifold; stationary, axisymmetric. SdS is special case $\mathfrak{a} = 0$.

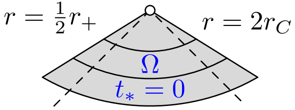
# Geometry of Schwarzschild–de Sitter black holes

$$g = -F(r)\,\mathrm{d}t^2 + F(r)^{-1}\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}.$$

Conformal embedding into $(\mathbb{R}^2, -\mathrm{d}z_0^2 + \mathrm{d}z_1^2)$ ('Penrose diagram'):

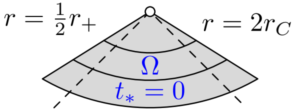



Better coordinates: $t_* = t - T(r)$, $T \sim |\log F|$ near $r = r_+, r_C$.
Metric extends (analytically) across horizons.




$$\Omega = [0,\infty)_{t_*} \times X, \qquad X = [\tfrac{1}{2}r_+, 2r_C] \times \mathbb{S}^2.$$

# Linear waves on SdS and KdS spacetimes




$$\Omega = [0,\infty)_{t_*} \times X, \qquad X = [\tfrac{1}{2}r_+, 2r_C] \times \mathbb{S}^2.$$

Initial value problem for the wave equation on $(\Omega, g_{\Lambda,\mathfrak{m},\mathfrak{a}})$:

$$\begin{cases} \Box_{g_{\Lambda,\mathfrak{m},\mathfrak{a}}}\phi = 0, \\ (\phi, \partial_{t_*}\phi)|_{t_*=0} \in \mathcal{C}^\infty(X) \oplus \mathcal{C}^\infty(X). \end{cases}$$

Theorem (authors on the next slide...)

*The solution $\phi$ has an asymptotic expansion:*

$$\phi(t_*,x) \sim \sum e^{-i\omega_j t_*} a_j(x), \qquad t_* \to \infty. \quad \textit{(Ignoring multiplicities.)}$$

*That is,* $|\phi(t_*,x) - \sum_{\operatorname{Im}\omega_j \geq -C} e^{-i\omega_j t_*} a_j(x)| \lesssim e^{-Ct_*}$ *for any* $C$.

# Resonance expansions on Kerr–de Sitter

## Theorem (2000s–2021)

*If $\phi$ solves $\Box_{g_{\Lambda,\mathfrak{m},\mathfrak{a}}}\phi = 0$ with smooth initial data, then*

$$\phi(t_*, x) \sim \sum e^{-i\omega_j t_*} a_j(x), \qquad t_* \to \infty.$$

- Bony–Häfner '08 (SdS case: $\mathfrak{a} = 0$), using Sá Barreto–Zworski '98 (information about $\omega_j$ when $\mathfrak{a} = 0$). See also Sá Barreto–Melrose–Vasy '09, '14.
- Dyatlov '11–'13 (slowly rotating KdS: $|\mathfrak{a}| \ll \mathfrak{m}$)
- Vasy '13 (not too fast rotating KdS: $|\mathfrak{a}| < \frac{\sqrt{3}}{2}\mathfrak{m}$, fixed $C > 0$)
- Petersen–Vasy '21 (full subextremal range, fixed $C > 0$)

$\mathrm{QNM}(\Lambda, \mathfrak{m}, \mathfrak{a}) := \{\omega_j\}$: set of resonances/quasinormal modes,

$$\mathrm{QNM}(\Lambda, \mathfrak{m}, \mathfrak{a}) = \{\omega \in \mathbb{C} \colon \exists a \in \mathcal{C}^\infty(X),\ \Box_{g_{\Lambda,\mathfrak{m},\mathfrak{a}}}(e^{-i\omega t_*} a(x)) = 0\}.$$

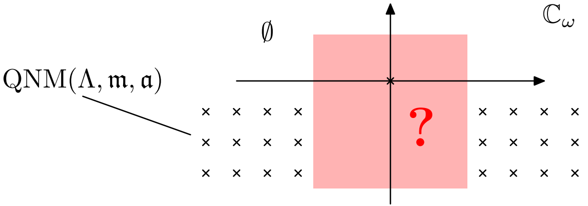
# Quasinormal modes of Kerr–de Sitter spacetimes

Subextremal KdS spacetime, parameters $\Lambda > 0$, $\mathfrak{m} > 0$, $|\mathfrak{a}| \lesssim \mathfrak{m}$.

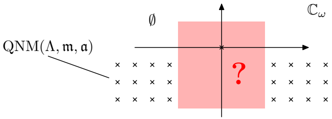
$$\phi(t_*, x) \sim \sum e^{-i\omega_j t_*} a_j(x), \qquad \mathrm{QNM}(\Lambda, \mathfrak{m}, \mathfrak{a}) := \{\omega_j\}$$

- Mode stability: $\omega_j = 0$ or $\operatorname{Im} \omega_j < 0$. Easy for $\mathfrak{a} = 0$. Known for $|\mathfrak{a}| \ll \mathfrak{m}$ (via perturbation theory)—see Dyatlov, Vasy.
- High energy regime $|\operatorname{Re} \omega| \gg 1$:
  - Sá Barreto–Zworski: $\omega_{ln} \approx (2l+1-i(n+\frac{1}{2}))\frac{(1-9\Lambda\mathfrak{m}^2)^{1/2}}{2\sqrt{27}\mathfrak{m}}$, $l \gg 1$, $n \in \mathbb{N}$. QNMs lie approximately on a lattice.
  - Dyatlov '13: asymptotic distribution of QNMs for $|\mathfrak{a}| \ll \mathfrak{m}$.

In general: at most finitely many in $\operatorname{Im} \omega \geq 0$—Petersen–Vasy.



# QNMs of Kerr–de Sitter spacetimes




## Why should one care?

- QNMs with $\operatorname{Im}\omega > 0$ are disastrous for nonlinear problems (including nonlinear stability).
- ✿✿✿ dinner with Maciej Zworski (2017 review article)

### Theorem (H., 2021)

*Fix $\Lambda > 0$ and $|\mathfrak{a}/\mathfrak{m}| < 1$. When $\mathfrak{m} > 0$ is sufficiently small:*

- *Mode stability holds for KdS black holes;*
- *QNMs in $\operatorname{Im}\omega > -C$ approximately lie in the set $-i\sqrt{\Lambda/3}\,\mathbb{N}_0$. (Convergence to this set, and convergence of mode solutions, as $\mathfrak{m} \searrow 0$.)*

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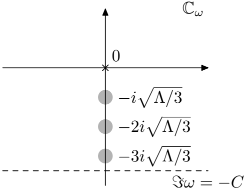
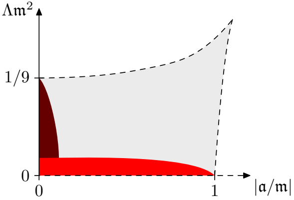
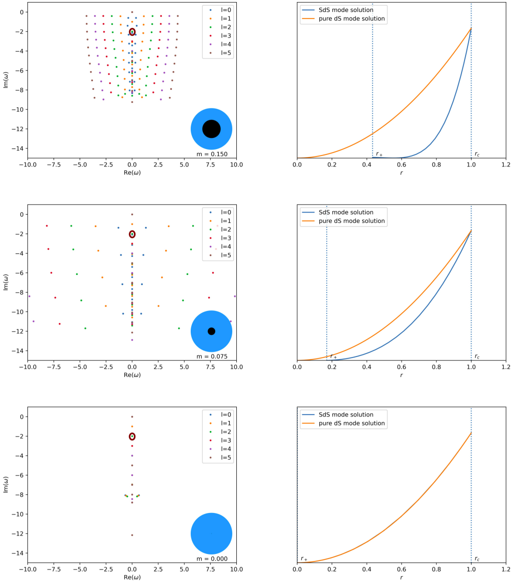


# Numerics for QNMs, $|\mathfrak{a}| = 0$ [H.–Xie 2021]



# The singular limit $\mathfrak{m} \searrow 0$: I

Recall: Schwarzschild–de Sitter metric (for $\Lambda = 3$)

$$g_{\Lambda,\mathfrak{m},0} = -\left(1 - \frac{2\mathfrak{m}}{r} - r^2\right) \mathrm{d}t^2 + \left(1 - \frac{2\mathfrak{m}}{r} - r^2\right)^{-1} \mathrm{d}r + r^2 g_{\mathbb{S}^2},$$

horizons at $r_+ \simeq 2\mathfrak{m}$ and $r_C \simeq 1$.

Limit #1: $\mathfrak{m} \searrow 0$ for fixed $r > 0$:

- limit is the de Sitter metric

$$g_{\Lambda,\mathrm{dS}} = -(1 - r^2)\,\mathrm{d}t^2 + (1 - r^2)^{-1}\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}.$$

- the black hole has completely disappeared: $g_{\Lambda,\mathrm{dS}}$ is smooth across $r = 0$!
- Set of quasinormal modes is $-i\sqrt{\Lambda/3}\mathbb{N}_0$.

## The singular limit $\mathfrak{m} \searrow 0$: II

Recall: Schwarzschild–de Sitter metric (for $\Lambda = 3$)

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Limit #2: $\hat{r} := r/\mathfrak{m}$, $\hat{t} := t/\mathfrak{m}$. Take $\mathfrak{m} \searrow 0$ for fixed $\hat{r}, \hat{t}$:

- limit of $\mathfrak{m}^{-2} g_{\Lambda,\mathfrak{m},0}$ is mass 1 Schwarzschild spacetime

  $$\hat{g} = -\Big(1 - \frac{2}{\hat{r}}\Big)\,\mathrm{d}\hat{t}^2 + \Big(1 - \frac{2}{\hat{r}}\Big)^{-1}\,\mathrm{d}\hat{r}^2 + \hat{r}^2 g_{\mathbb{S}^2}.$$

  Event horizon at $\hat{r} = 2$. Cosmological horizon has disappeared; instead, have asymptotically flat infinity.
- $e^{-i\omega t} = e^{-i(\mathfrak{m}\omega)\hat{t}}$. Note $\operatorname{Im}\omega \geq -C \Rightarrow \operatorname{Im}(\lim_{\mathfrak{m}\searrow 0} \mathfrak{m}\omega) \geq 0$.
- in Kerr–de Sitter case ($\hat{\mathfrak{a}} = \mathfrak{a}/\mathfrak{m}$ fixed), the limit is the Kerr spacetime (mass 1, angular momentum $\hat{\mathfrak{a}}$). Mode stability known (Whiting '89, Shlapentokh-Rothman '15, Casals–Teixeira da Costa '21).

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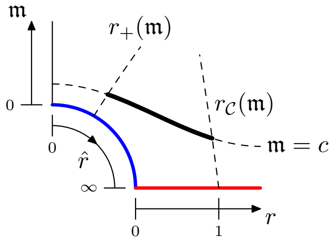
# Geometry and analysis in the singular limit $\mathfrak{m} \searrow 0$

For fixed $\omega$, work on total space $[[0,1)_{\mathfrak{m}} \times [0,2)_r \times \mathbb{S}^2; \{\mathfrak{m}=r=0\}]$.




- Study spectral family $\widehat{\Box_{g_{\mathfrak{m}}}}(\omega) = e^{i\omega t_*}\Box_{g_{\mathfrak{m}}} e^{-i\omega t_*}$ ($g_{\mathfrak{m}}$: KdS metric with fixed $\Lambda$ and specific angular momentum $\mathfrak{a}/\mathfrak{m}$).
- Prove uniform a priori estimates for $\widehat{\Box_{g_{\mathfrak{m}}}}(\omega)u = f$.
  - Use invertibility of Schwarzschild/Kerr model, and
  - use invertibility of de Sitter model (away from its QNMs),

  to prove injectivity of $\widehat{\Box_{g_{\mathfrak{m}}}}(\omega)$ for small $\mathfrak{m}$ (on function spaces adapted to the singular limit).
- Delicate caveat: analytically, the de Sitter model has a *conic singularity* where the black hole used to be!

# Outlook

- Prove analogue of the main Theorem for other equations of interest (Teukolsky, Maxwell, linearized Einstein).
- Prove nonlinear stability of Kerr–de Sitter black holes in the 'almost full subextremal range' covered by the Theorem.

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Thank you for your attention!