# Pseudo-homogeneous Lagrangian manifolds and asymptotics in the Cauchy problem with localized initial data

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# Model problem

Potential motion of a liquid in the uniform field of gravity in a horizontally infinite basin of finite depth

Parameters:

- Basin depth $d$
- Characteristic basin horizontal size $L$
- Initial perturbation horizontal size $l$
- Initial perturbation vertical amplitude $A$

**Assumptions:**

$$\mu = d / L, \quad l / L, \quad A / d \ll 1$$

$d \sim l$ (strong dispersion case)

**Equation for the free surface elevation:**

$$\mu^2 \frac{\partial^2 \eta}{\partial t^2} + \widehat{\mathcal{H}}\eta = 0, \qquad \eta|_{t=0} = V\left(\frac{x}{\mu}\right), \qquad \eta_t|_{t=0} = 0$$

$$\widehat{\mathcal{H}} = \mathcal{H}\left(\overset{2}{x}, -i\mu\overset{1}{\frac{\partial}{\partial x}}, \mu\right), \quad \mathcal{H}(x,p,\mu) = H(x,p) - \frac{ih}{2}\sum_{j=1}^{2} H_{x_j p_j}(x,p) + O(\mu^2),$$

$$H(x,p) = |p| \tanh(D(x)|p|)$$

Dobrokhotov, Zhevandrov, Funct. Anal. Appl. 1985

# Cauchy problem

**Evolution equation** with small parameter $\mu > 0$

$$\mathcal{H}\left(\overset{2}{t},\overset{2}{x},-i\mu\overset{1}{\frac{\partial}{\partial t}},-i\mu\overset{1}{\frac{\partial}{\partial x}},\mu\right)\vec{\Psi}(x,t,\mu)=0, \quad t\in[0,T], \quad x\in\mathbb{R}^n,$$

where the symbol $\mathcal{H}(t,x,p_0,p,\mu)$ is an $r\times r$ matrix and a polynomial of degree $l$ in $p_0$

**Localized initial conditions**

$$\frac{\partial^j\vec{\Psi}}{\partial t^j}(0,x,\mu)=\vec{V}_j\left(\frac{x'-f(x'')}{\mu},x''\right), \quad j=0,1,\dots,l-1, \qquad \vec{V}_j(y',x'') \text{ decay as } y'\to\infty$$

near the submanifold $Y=\{x'=f(x'')\}\subset\mathbb{R}^n, \quad x'=(x_1,\dots,x_k), \quad x''=(x_{k+1},\dots,x_n)$

Task: construct an asymptotic solution as $\mu\to 0$

(under certain conditions on the symbol $\mathcal{H}$)

# Representation of localized initial data

- Simplest case: $Y = \{x_0\}$ is a point

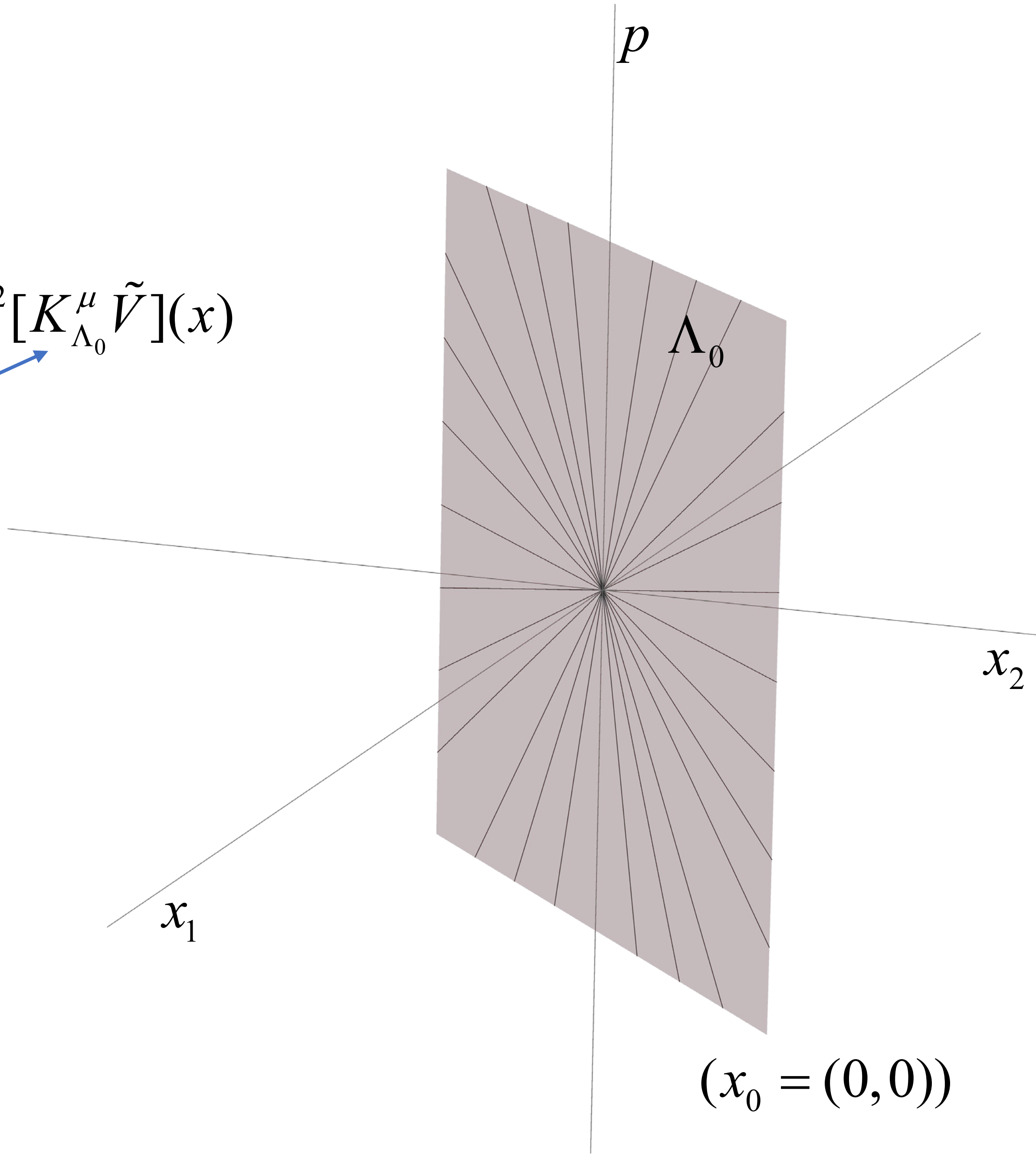
$$V\left(\frac{x-x_0}{\mu}\right) = \left(\frac{i}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{\frac{i}{\mu}\langle p,x\rangle} \tilde{V}(p)\,dp = \mu^{n/2}[K^{\mu}_{\Lambda_0}\tilde{V}](x)$$

Maslov's canonical operator
on the Lagrangian manifold

$$\Lambda_0 = \{(x,p) \in \mathbb{R}^{2n} : x = x_0\}$$

with the measure $d\sigma = dp_1 \wedge \ldots \wedge dp_n$

$\tilde{V}$ is the Fourier transform of $V$



## Representation of localized initial data (continued)

- General case: $Y$ is a submanifold of codimension $\operatorname{codim} Y = k$

$$V\left(\frac{x'-f(x'')}{\mu}, x''\right) = \left(\frac{i}{2\pi}\right)^{k/2} \int_{\mathbb{R}^k} e^{\frac{i}{\mu}\langle p', x'-f(x'')\rangle} \tilde{V}(p', x'')\,dp' = \mu^{k/2}[K^{\mu}_{\Lambda_0}\tilde{V}](x)$$

partial Fourier transform

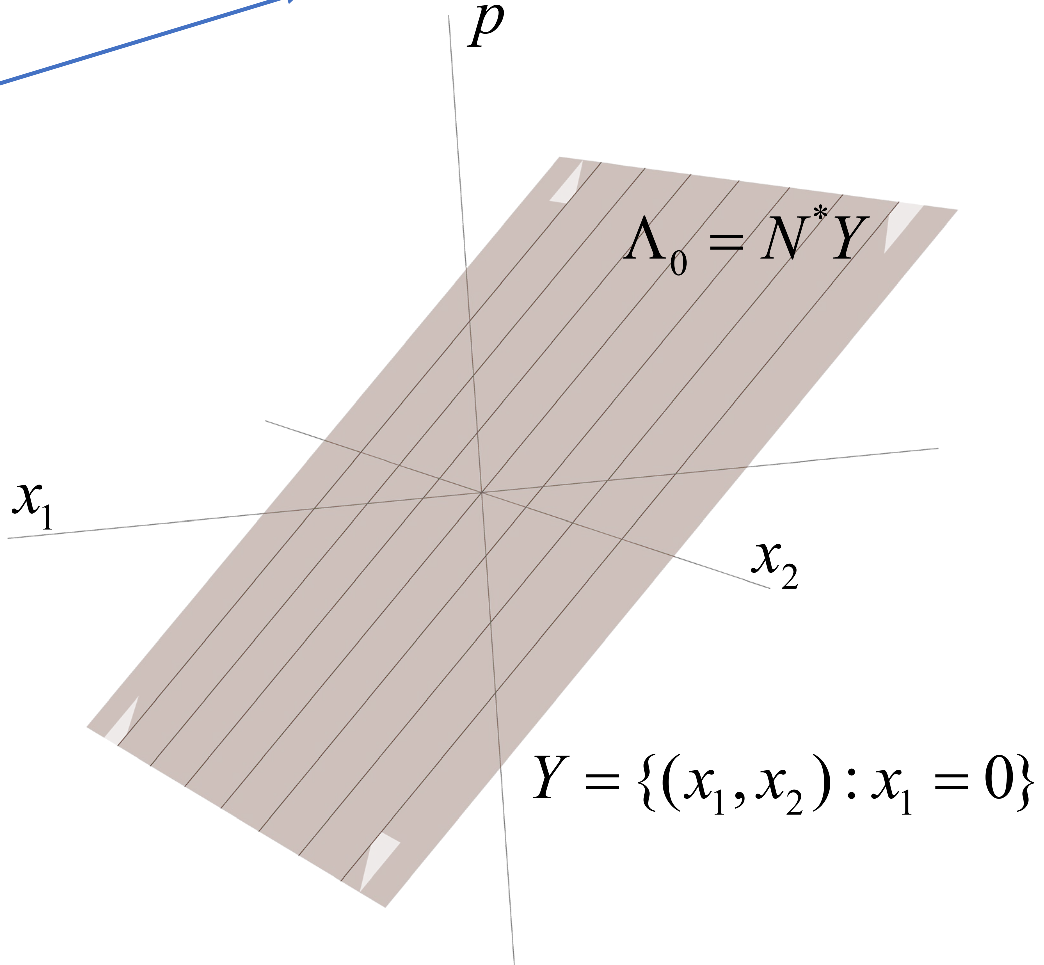
Maslov's canonical operator on the Lagrangian manifold

$$\Lambda_0 = \{(x,p) \in \mathbb{R}^{2n} : x \in Y, \langle p, \xi\rangle = 0 \ \forall \xi \in T_x Y\}$$

with the measure

$$d\sigma = dp_1 \wedge \ldots \wedge dp_k \wedge dx_{k+1} \wedge \ldots \wedge dx_n$$

$\Lambda_0 = N^* Y$ is the conormal bundle of $Y$



# General scheme for constructing semiclassical asymptotics in the Cauchy problem

$$\mathcal{H}\left(\overset{2}{t},\overset{2}{x},-i\mu\overset{1}{\frac{\partial}{\partial t}},-i\mu\overset{1}{\frac{\partial}{\partial x}},\mu\right)\vec{\Psi}(x,t,\mu)=0,$$

$\mathcal{H}(t,x,p_0,p,\mu)$ is an $r\times r$ matrix and a polynomial of degree $l$ in $p_0$

$$\frac{\partial^j\vec{\Psi}}{\partial t^j}(0,x,\mu)=[K^{\mu}_{\Lambda_0}\vec{A}_j](x),\quad j=0,1,\ldots,l-1,$$

$\Lambda_0$ is a Lagrangian manifold in $\mathbb{R}^{2n}_{(x,p)}$
$K^{\mu}_{\Lambda_0}$ is Maslov's canonical operator

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**Solution:** *effective Hamiltonians* $\lambda_s(t,x,p)$
(roots of the equation $\det\mathcal{H}(t,x,\lambda,p,0)=0$)

$$\Lambda^s_t=g^t_{\lambda_s}(\Lambda_0)$$

shifts along the trajectories of Hamiltonian systems

$$\vec{\Psi}(x,t,\mu)=\sum_s[K^{\mu}_{\Lambda^s_t}\vec{A}_s(t)](x)$$

# Difficulties in specific problems. Examples of effective Hamiltonians

The general scheme works "as is" if the roots are real and smooth (of constant multiplicity) at least in a neighborhood of the Lagrangian manifolds etc. etc. etc.

The roots of the equation $\det \mathcal{H}(t,x,\lambda,p,0)=0$ are branches of an algebraic function of the coefficients $\Longrightarrow$ nonsmoothness (at the branching points)

One often has nonsmoothness at $p=0$

**Examples:**

- Petrovsky hyperbolic systems: $\lambda_s(t,x,p)$ are homogeneous functions of $p$
- Wave equation: just the same, $\lambda_\pm(x,p)=\pm c(x)|p|$
- Water wave equation with dispersion: $\lambda_\pm(x,p)=\pm\sqrt{|p|\tanh(|p|D(x))}$
- Wave equation on a 2D lattice: $\lambda_\pm(x,p)=\pm c(x)\sqrt{\sin^2 p_1+\sin^2 p_2}$

# A class of effective Hamiltonians: – *pseudo-homogeneous Hamiltonians*

That is, Hamiltonians admitting **expansion in homogeneous functions**:

$$\lambda(x,p)=\lambda^{(1)}(x,p)+\lambda^{(2)}(x,p)+\ldots+\lambda^{(N)}(x,p)+O(|p|^{N+1}),\quad p\to 0,\quad \forall N$$

$$\lambda^{(j)}(x,\tau p)=\tau^{j}\lambda(x,p),\quad \tau>0$$

smooth for $p\neq 0$

or, equivalently,

$\lambda(x,r\omega)=:\lambda(x,r,\omega)$ is a smooth functions of $x\in\mathbb{R}^n,\ r\in[0,\infty),\ \omega\in\mathbb{S}^{n-1}$

$$\lambda(x,0,\omega)=0$$

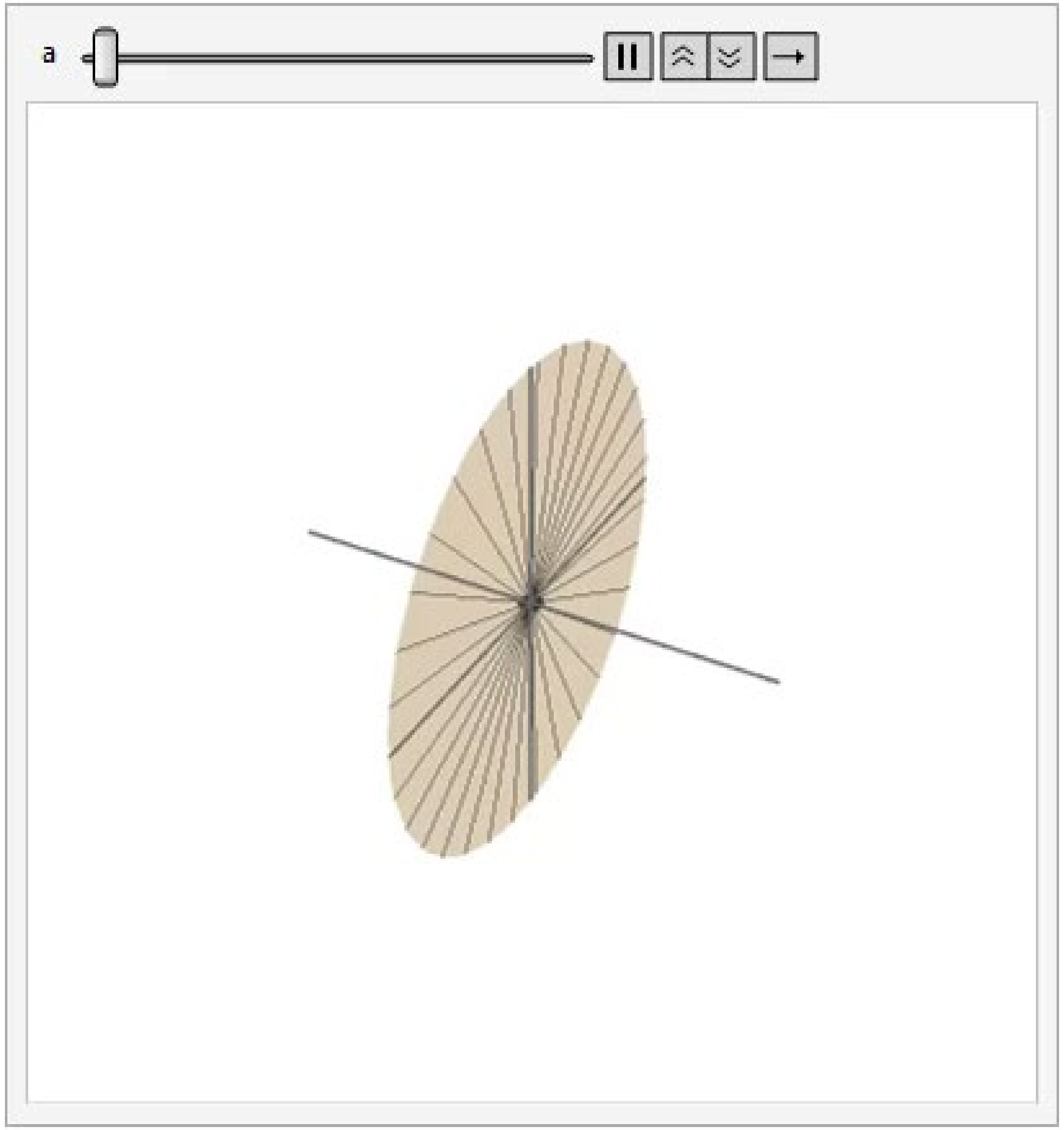
The substitution

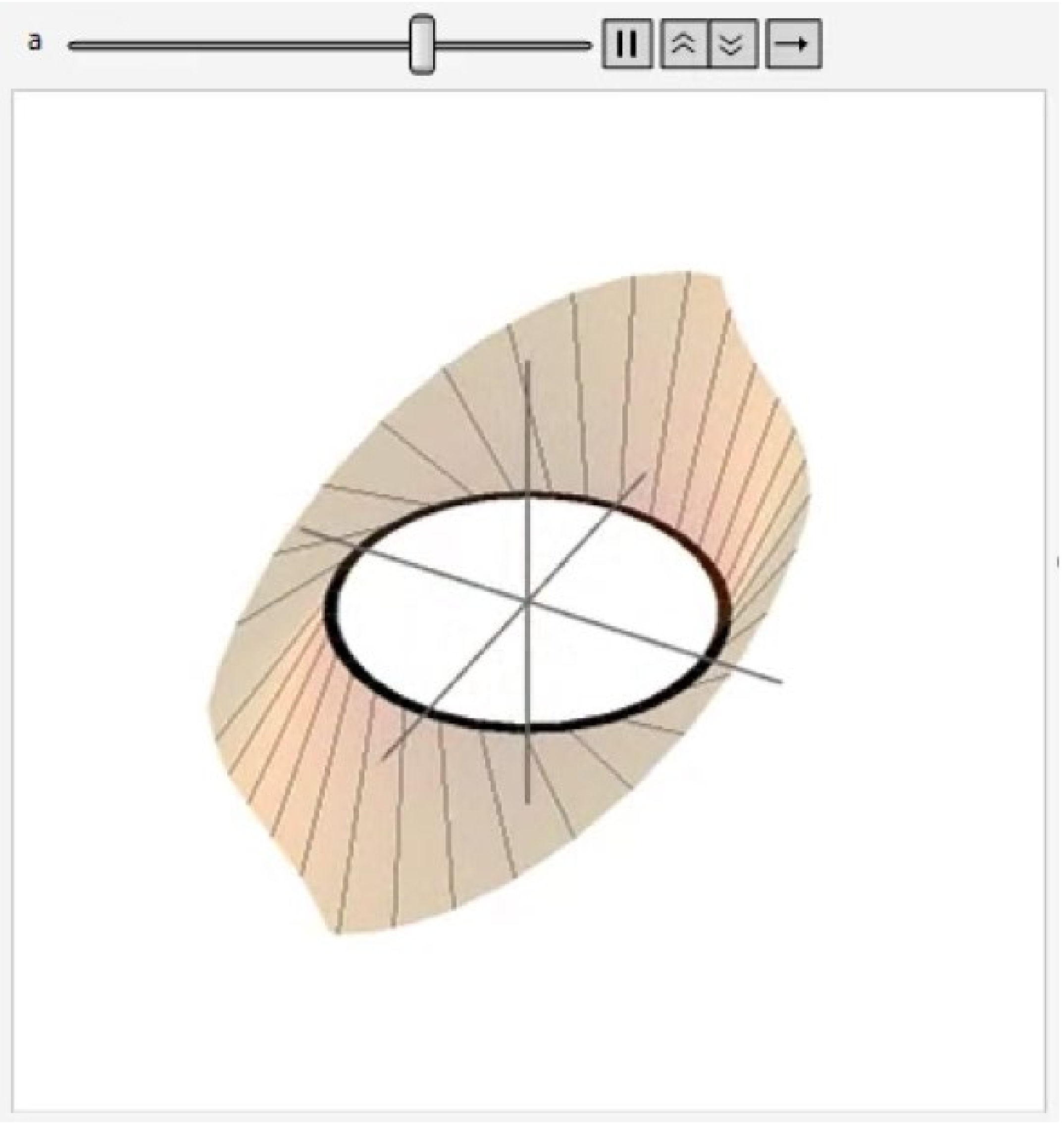
$$p=r\omega,\quad r\in[0,\infty),\ \omega\in\mathbb{S}^{n-1}$$

is very convenient

# What does the Hamiltonian flow do with the Lagrangian manifold?

"Punctured" manifold



# Punctured Lagrangian manifolds

$\Lambda$ is a smooth $n$-dimensional manifold with boundary $\partial\Lambda$

$$\Lambda \ni \alpha = (\text{near the boundary}) = (\phi, \rho), \quad \rho \geq 0, \quad \partial\Lambda = \{\rho = 0\}$$

**Definition**

*A punctured Lagrangian manifold* := a smooth mapping $\jmath : \Lambda \to \mathbb{R}^{2n}$,

$$\alpha \mapsto (X(\alpha), P(\alpha))$$

- One has $P_\alpha^T X_\alpha - X_\alpha^T P_\alpha = 0 \quad (\jmath^*(dp \wedge dx) = 0)$ (Lagrangian property)
- For $\alpha \notin \partial\Lambda$: $P(\alpha) \neq 0, \quad \operatorname{rank}\begin{pmatrix} X_\alpha \\ P_\alpha \end{pmatrix} = n$ immersion
- For $\alpha = (\phi, 0) \in \partial\Lambda$: $\operatorname{rank}\begin{pmatrix} X_\phi(\phi,0) & 0 \\ P_{\phi\rho}(\phi,0) & P_\rho(\phi,0) \end{pmatrix} = n$

The last condition $\Leftrightarrow \mathbf{\Lambda} := \partial\Lambda \times \mathbb{R}_+ \ni (\phi, \tau) \mapsto (X(\phi,0), \lim_{\varepsilon\to 0} \varepsilon^{-1} P(\phi, \varepsilon\tau))$

is a homogeneous Lagrangian manifold

Invariance with respect to the Hamiltonian flow

**Theorem**

The class of punctured Lagrangian manifolds is invariant under shifts along Hamiltonian vector fields corresponding to pseudo-homogeneous Hamiltonians.

# Statement of the problem

On punctured Lagrangian manifolds, define a canonical operator

$$K^{\mu}_{\Lambda} : C_0^{\infty}(\Lambda) \longrightarrow H^{\infty}_{\mu}(\mathbb{R}^n)$$

$$H^{\infty}_{\mu}(\mathbb{R}^n) = \bigcap_{s=0}^{\infty} H^{s}_{\mu}(\mathbb{R}^n), \qquad \|u\|_s = \sup_{\mu\in(0,1)} \left\|(1-\mu^2\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)} .$$

On functions in $C_0^{\infty}(\Lambda^{\circ}) \subset C_0^{\infty}(\Lambda)$ , where $\Lambda^{\circ} = \Lambda \setminus \partial\Lambda$ , it must

Coincide with the “standard” Maslov canonical operator.

# Microlocal definition of the standard canonical operator

following [Izv17]

*Nondegenerate phase function*: $\Phi(x,\xi)$ is defined on an open set $V \subset \mathbb{R}^n_x \times \mathbb{R}^m_\xi$

The differentials $d(\Phi_{\xi_1}), \ldots, d(\Phi_{\xi_m})$ are linearly independent on

$$C_\Phi = \{(x,\xi) \in V : \Phi_\xi(x,\xi) = 0\}$$

$\Lambda_\Phi = j_\Phi(C_\Phi)$ is a Lagrangian mfld

$$j_\Phi : (x,\xi) \mapsto (x, \Phi_x(x,\xi))$$

$\Lambda_\Phi \subset \Lambda$ $\quad$ $\Phi$-chart on $\Lambda$

*Pre-canonical operator in the $\Phi$-chart:*

$$[K^\mu_\Phi A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \int_V e^{\frac{i}{\mu}\Phi(x,\xi)} a(x,\xi)\, d\xi_1 \cdots d\xi_m$$

cf. Hörmander's Fourier integral distributions

$$A \in C_0^\infty(\Lambda) \qquad a|_{C_\Phi} = j_\Phi^*(A)\sqrt{F_{\Phi,d\sigma}},$$

$$F_{\Phi,d\sigma} = \frac{j_\Phi^*(d\sigma) \wedge d(-\Phi_{\xi_1}) \wedge \ldots \wedge d(-\Phi_{\xi_m})}{dx_1 \wedge \ldots \wedge dx_n \wedge d\xi_1 \wedge \ldots \wedge d\xi_m}$$

$d\sigma$ is a measure (volume form) on $\Lambda$

# New phase functions

*We need a new notion of phase function to describe a punctured Lagrangian manifold near the boundary*

$V \subset \mathbb{R}^n_x \times \mathbb{R}^{m-1}_\theta \times \overline{\mathbb{R}}_{+\rho}$ is a (relatively) open set

$\Phi(x,\theta,\rho)$ is a smooth real function on $V$, $\Phi(x,\theta,0) = 0$

Define

$$\Psi(x,\theta,\rho) = \frac{1}{\rho}\Phi(x,\theta,\rho)$$

The following conditions hold on the set

$$C_\Phi = \{(x,\theta,\rho) \in V : \Psi_\theta(x,\theta,\rho) = 0, \quad \Psi(x,\theta,\rho) + \rho\Psi_\rho(x,\theta,\rho) = 0\}$$

| $\rho > 0$ | $\rho = 0$ |
|---|---|
| (i) $\Phi_x \neq 0$<br>(ii) $\operatorname{rank}\begin{pmatrix}\Phi_{\xi x}(x,\xi) & \Phi_{\xi\xi}(x,\xi)\end{pmatrix} = m$<br>$\xi = (\theta,\rho)$ | $\operatorname{rank}\begin{pmatrix}\Psi_{\theta x}(x,\theta,0) & \Psi_{\theta\theta}(x,\theta,0) \\ \Psi_x(x,\theta,0) & 0\end{pmatrix} = m$ |

# New phase functions (continued)

**Теорема**

Punctured Lagrangian manifolds = Lagrangian manifolds locally described by phase functions of this form via the mapping

$$\jmath_{\Phi} : (x, \xi) \mapsto (x, \Phi_x(x, \xi))$$

# What about oscillatory integrals?

First, consider the initial function

$$I(x,\mu)=\left(\frac{i}{2\pi\mu}\right)^{k/2}\int_{\mathbb{R}^k} e^{\frac{i}{\mu}\langle p',x'-f(x'')\rangle}\tilde{V}(p',x'')dp',$$

localized on a submanifold $Y,\ \operatorname{codim} Y = k.$
The corresponding Lagrangian manifold is the conormal bundle $\Lambda_0 = N^*Y$
with natural coordinates $(p',x'')=(p_1,\ldots,p_k,x_{k+1},\ldots,x_n)$ and with
measure $d\sigma = dp_1\wedge\ldots\wedge dp_k\wedge dx_{k+1}\wedge\ldots\wedge dx_n$ homogeneous of degree $k$ in $p'$.

In the polar coordinates:

$$I(x,\mu)=\left(\frac{i}{2\pi\mu}\right)^{k/2}\int_0^\infty d\rho\int d\theta\, e^{\frac{i\rho}{\mu}\langle\theta,x'-f(x'')\rangle}\rho^{k-1}\tilde{V}(\rho\theta,x''),\ \ \theta\rho = p'$$

The degree of homogeneity of the measure=codimension

# What about oscillatory integrals? (continued)

*Pre-canonical operator in the $\Phi$ -chart*

$\Phi = \rho\Psi$

$$[K^{\mu}_{\Phi}A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}}\int_0^\infty d\rho \int d\theta\, e^{\frac{i\rho}{\mu}\Psi(x,\theta,\rho)} a(x,\theta,\rho), \qquad A \in C_0^\infty(\Lambda),$$

$$a|_{C_\Phi} = \jmath_\Phi^*(A)\sqrt{F_{\Phi,d\sigma}}.$$

$$F_{(\Phi,d\sigma)} = \frac{\jmath_\Phi^*(d\sigma)\wedge d(-\Phi_{\theta_1})\wedge\cdots\wedge d(-\Phi_{\theta_{m-1}})\wedge d(-\Phi_\rho)}{dx_1\wedge\cdots\wedge dx_m\wedge d\theta_1\wedge\ldots d\theta_{m-1}\wedge d\rho} \qquad \operatorname{ord} F_{(\Phi,d\sigma)} = k+m-2$$

Hence $a(x,\theta,\rho) = \rho^{\frac{k+m}{2}-1} b(x,\theta,\rho), \quad b\in C_0^\infty.$

$\Psi^\rho(x,\theta) = \Psi(x,\theta,\rho)$ is a nondegenerate phase function on its own

$$[K^{\mu}_{\Phi}A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}}\int_0^\infty \rho^{\frac{k+m}{2}-1}[K^{\mu/\rho}_{\Lambda_{\Psi^\rho}}B^\rho]\left(x,\frac{\mu}{\rho}\right)d\rho, \quad B^\rho \in C_0^\infty(\Lambda_{\Psi^\rho}),$$

$h = \dfrac{\mu}{\rho}$ new small parameter

## What about oscillatory integrals? (still continued)

**Thus, what we have is this:**

$$[K_\Phi^\mu A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \int_0^\infty \rho^{\frac{k+m}{2}-1} [K_{\Lambda_{\Psi^\rho}}^{\mu/\rho} B^\rho]\left(x, \frac{\mu}{\rho}\right) d\rho, \quad B^\rho \in C_0^\infty(\Lambda_{\Psi^\rho}),$$

$h = \dfrac{\mu}{\rho}$ new small parameter

If we wish to consider lower-order terms (and we must!), we need

(i) $B^\rho = B^\rho(\cdot, h) = B_0^\rho + hB_1^\rho + h^2 B_2^\rho + \ldots$ (ii) replace the integration interval by $[c, +\infty), \quad c > 0.$

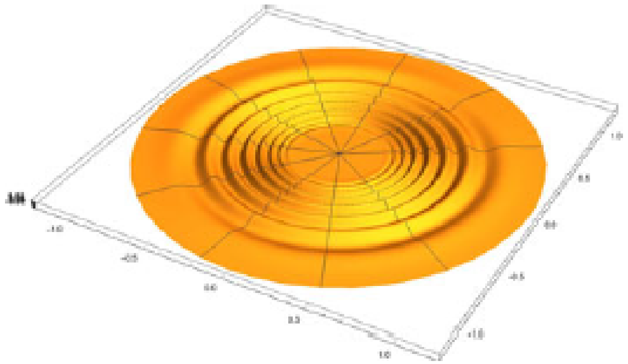
**Typical asymptotic expansion:** $$\frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \sum_{j=1}^\infty \int_c^\infty \rho^{\frac{k+m}{2}-1} \left(\frac{\mu}{\rho}\right)^j [K_{\Lambda_{\Psi^\rho}}^{\mu/\rho} B_j^\rho]\left(x, \frac{\mu}{\rho}\right) d\rho, \quad B_j^\rho \in C_0^\infty(\Lambda_{\Psi^\rho})$$

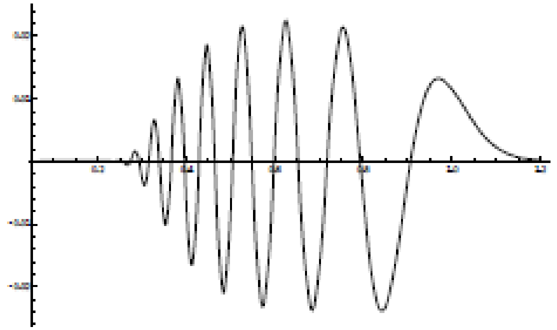
**Maximum possible accuracy:** $$R_{(c,d)j} = \frac{1}{\mu^{m/2}} \int_{c\mu}^{d\mu} \rho^{\frac{k+m}{2}-1} \left(\frac{\mu}{\rho}\right)^j f_j(x,\rho)\, d\rho$$

$$\| \nabla^\beta R_{(c,d)j} \| \le C\mu^{k/2}$$

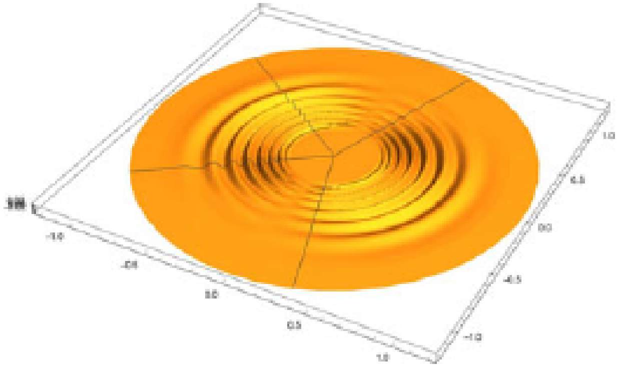
# Examples of solution of the original problem

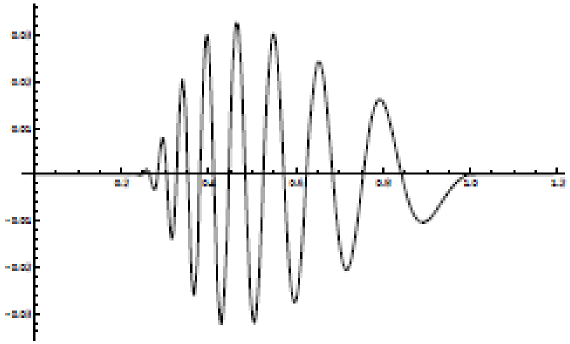
$$\mu^2 \frac{\partial^2 \eta}{\partial t^2} + \widehat{\mathcal{H}}\eta = 0, \qquad \eta\,|_{t=0} = V\left(\frac{x}{\mu}\right), \qquad \eta_t\,|_{t=0} = 0, \qquad H(x,p) = |p|\tanh(D(x)\,|p|)$$





$$t = 1, \quad \mu = 0.02, \quad V(y) = e^{-|y|^2/2}$$





$$t = 1, \quad \mu = 0.02, \quad V(y) = y_1 e^{-|y|^2/2}$$

# Literature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Thank you!