Weighted Zeta Functions for Hyperbolic Flows

(joint work with T. Weich)

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- $\varphi_t \curvearrowright S\mathbf{X}$ the geodesic flow, V its generator (geodesic vector field)

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Theorem (Dyatlov and Guillarmou 2016)

The L^2*-resolvent* $\mathbf{R}(\lambda) = (V - \lambda)^{-1}$ *continues meromorphically to* $\mathbb{C}$ *as a family of operators* $\mathrm{C}^\infty(S\mathbf{X}) \to \mathcal{D}'(S\mathbf{X})$*.*

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The poles of $\mathbf{R}(\lambda)$ are called the Ruelle resonances of V. The image $\mathrm{range}(\Pi_{\lambda_0})$ is called the space of resonant states.

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- we are looking for means of calculating $\mathcal{T}_{\lambda_0}$ concretely

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- generalization of dynamical determinants used to prove meromorphic continuation of the Ruelle zeta function (Dyatlov and Zworski 2016; Dyatlov and Guillarmou 2016)

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- holds in a vector-valued version

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which makes the *flat trace* well-defined:

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(3) use (weighted) Guillemin trace formula to show that

$$Z_f(\lambda) = \mathrm{e}^{-\lambda t_0}\mathrm{tr}^\flat\left(\mathrm{e}^{-t_0 V}\mathbf{R}(\lambda)f\right)$$

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Definition

We call the following distribution on $S\mathbf{X}$ the Wigner distribution associate with φ_i:

$$\mathrm{W}_{\varphi_i}[f] := \langle \mathrm{Op}(f)\varphi_i, \varphi_i \rangle_{\mathrm{L}^2} \ .$$

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- holds more generally for compact locally symmetric spaces of rank one

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- allows us to compute quantum mechanical objects in terms of classical quantities
- uses an exact correspondence between invariant Ruelle and so-called Patterson-Sullivan distributions obtained by (Guillarmou, Hilgert, and Weich 2021)
- extends results by (Anantharaman and Zelditch 2007) and (Emonds 2014) beyond the hyperbolic setting and to general smooth f

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- use techniques adapted from (Borthwick 2014) for the actual computations

Some Example Plots

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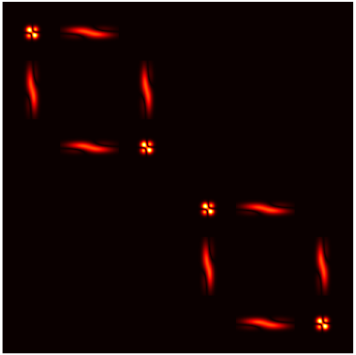
Figure: Invariant Ruelle distribution on a Poincaré section $\Sigma \subseteq S\mathbf{X}$ of the unit tangent bundle of the symmetric three-funnel surface of length 14 associated with a resonance near the leading one.



More Example Plots

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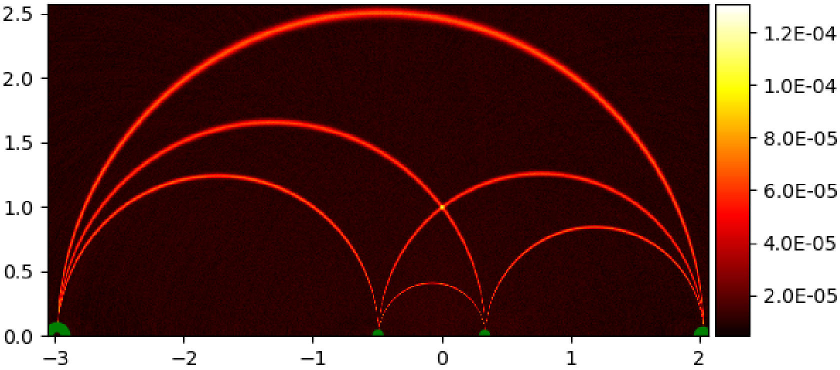
Figure: Invariant Ruelle distribution of the symmetric three-funnel surface of length 14 associated with a resonance near the leading one and pushed forward along the canonical projection $S\mathbf{X} \to \mathbf{X}$.



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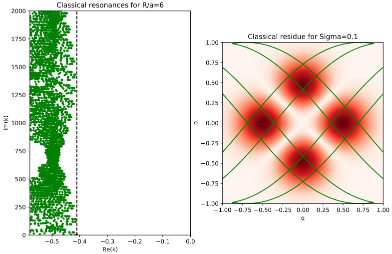

Thank you for your attention!

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Abstract

Abstract. Ruelle resonances constitute important invariants for chaotic (hyperbolic) dynamical systems and their theory has progressed greatly in the last couple of decades. Building on the work of Dyatlov and Guillarmou (2016) in this subject area we define and discuss a notion of weighted zeta function for open hyperbolic systems. First we sketch a proof of their meromorphic continuation and the fact that their poles encode the resonances. Then we show how their residues can be identified with so-called invariant Ruelle distributions. On the one hand this yields a residue interpretation of Patterson-Sullivan distributions, on the other hand this enables their numerical calculation for example systems like geodesic flows on Schottky surfaces and 3-disk obstacle scattering.