The self-interacting Dirac fields in FLRW spacetime

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- The Dirac equation is one of the important equations of relativistic quantum mechanics, quantum electrodynamics, and quantum field theory.
- The Dirac equation and its quantization in curved spacetime are of great interest due to the role of spin-$\frac{1}{2}$ particles in astrophysics and cosmology.
- Recent observational confirmation of the expansion of the universe and the quantum field theory demand a detailed investigation of the solutions of the Dirac equation in curved spacetime (Birrell & Davies, Parker & Toms and others).

- The construction of a quantum field theory in curved spacetime relies heavily on a detailed description of the solutions of the Dirac equation in a curved background. The standard models of Cosmology provide such backgrounds, which form a family of curved backgrounds under the name Friedmann-Lemaître-Robertson-Walker (FLRW) models.
- For the FLRW spacetime with accelerated expansion or contraction, more exactly, for the de Sitter spacetime Yagdjian'2020 gives a fundamental solution of the Dirac operator and obtained an explicit formula for the solution of the Cauchy problem.

- Several exact solutions of the Dirac equation in the FLRW spacetime, which is the basis of the standard cosmological model, obtained by the separation of variables approach answer to some very interesting questions of physics (see, Finster & Reintjes, de Oliveira& Schmidt, Zecca and others)
- The behavior of particles obeying the covariant Dirac equation in the FLRW spacetime has given rise to many investigations (Schrodinger, Barut & Duru, de Oliveira & Schmidt, Fock, Parker & Toms, Zecca).

- On the other hand, the solution of the semilinear Dirac equation describing self-interacting Dirac fields in general FLRW spacetime according to our best knowledge, haven't been discussed in literature.
- We study solutions of the semilinear Dirac equation in the curved spacetime of the FLRW models of cosmology.
- We focus on the relationship between the mass term, scale factor, nonlinear term, and initial function, which provides a global in time existence or an estimate on the lifespan of the solution of the Dirac equation in the expanding universe.

We consider the spatially flat FLRW models such that the metric tensor in Cartesian coordinates, is written as follows

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \quad \mu, \nu = 0, 1, 2, 3, \tag{1}$$

with $a(t) = a_0 t^{\ell}$, $\ell \in \mathbb{R}$, $t > 0$, and $x \in \mathbb{R}^3$, $x_0 = t$.
If $\ell < 0$ the spacetime is contracting.
If $\ell > 1$ the expansion is accelerating (with horizon).
If $0 < \ell < 1$ the expansion is decelerating.
If $\ell = 1$ is the Milne spacetime
Factor $a(t) = a_0 t^{2/3}$ is modeling the matter dominated universe
Factor $a(t) = a_0 t^{1/2}$ is modeling the radiation dominated universe

The curvature of the space with $a(t)=a_0t^\ell$ is

$$R(t)=-6\ell\,(1-2\ell)\,t^{-2}$$

The Dirac equation in the spacetime with the metric tensor (1) is

$$\left(i\gamma^0\partial_t+i\frac{1}{a(t)}\gamma^1\partial_{x_1}+i\frac{1}{a(t)}\gamma^2\partial_{x_2}+i\frac{1}{a(t)}\gamma^3\partial_{x_3}+i\frac{3\dot a(t)}{2a(t)}\gamma^0-m\mathbb{I}_4\right)\Psi=F\,, \tag{2}$$

where the contravariant gamma matrices are

$$\gamma^0=\begin{pmatrix}\mathbb{I}_2 & \mathbb{O}_2\\ \mathbb{O}_2 & -\mathbb{I}_2\end{pmatrix},\ \gamma^k=\begin{pmatrix}\mathbb{O}_2 & \sigma^k\\ -\sigma^k & \mathbb{O}_2\end{pmatrix},\ k=1,2,3\,;\gamma^5:=-i\gamma^0\gamma^1\gamma^2\gamma^3$$

Here σ^k are the Pauli matrices

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and $\mathbb{I}_n$, $\mathbb{O}_n$ denote the $n\times n$ identity and zero matrices.

Consider the Dirac equation

$$\left(i\gamma^0\partial_0 + i\frac{1}{a(t)}\sum_{k=1,2,3}\gamma^k\partial_{x_k} + i\frac{3\dot{a}(t)}{2a(t)}\gamma^0 - m\mathbb{I}_4 + \gamma^0 V(x,t)\right)\Psi = 0\,,$$

where V is the matrix-valued potential

$$V(x,t) : \mathbb{R}^3 \times (0,\infty) \longrightarrow M_4(\mathbb{C}).$$

The last equation can be written in the equivalent form

$$\left(\partial_0 + \frac{1}{a(t)}\sum_{j=1,2,3}\alpha^j\partial_j + \frac{3\dot{a}(t)}{2a(t)}\mathbb{I}_4 + im\gamma^0 - iV(x,t)\right)\Psi = 0\,,$$

where $\alpha^j = \gamma^0\gamma^j$ are self-adjoint matrices, $(\alpha^j)^* = \alpha^j$.

Condition ($\mathcal{L}$) *The function $F = F(\psi) \in C^3(\mathbb{C}^4; \mathbb{C}^4)$ is Lipschitz continuous with exponent α in the space $H_{(s)}(\mathbb{R}^3)$, that is, there is a constant $C > 0$ such that*

$$\|F(\psi_1) - F(\psi_2)\|_{H_{(s)}(\mathbb{R}^3)}$$
$$\leq C\|\psi_1 - \psi_2\|_{H_{(s)}(\mathbb{R}^3)} \left(\|\psi_1\|^{\alpha}_{H_{(s)}(\mathbb{R}^3)} + \|\psi_2\|^{\alpha}_{H_{(s)}(\mathbb{R}^3)}\right) .$$

The polynomial in ψ vector-valued functions and functions
$F(\psi) = (\gamma^0\psi, \psi)\gamma^0\psi$, $F(\psi) = |\gamma^0\gamma^5\psi|^{\alpha}\psi$,
$F(\psi) = |\gamma^0\gamma^5\psi|^{\alpha}\gamma^0\psi$, $F(\psi) = \pm|\psi|^{1+\alpha}$, $F(\psi) = \pm|\psi|^{\alpha}\psi$
are important examples of the Lipschitz continuous with exponent $\alpha > 0$ in the Sobolev space $H_{(s)}(\mathbb{R}^3)$, $s \geq 3$, functions.

We consider the Cauchy problem with the initial time $t = 1$.
We admit that the mass of the field (particle) can be changing in time and vanishing at infinity.
The model that we choose is determined by the Dirac operator

$$\mathscr{D}_{FLRW}(t, \partial_t, \partial_x) := i\gamma^0\partial_0 + i\frac{1}{a(t)}\sum_{j=1,2,3}\gamma^j\partial_{x_j} + i\frac{3\dot{a}(t)}{2a(t)}\gamma^0 - mt^{-1}\mathbb{I}_4\,, \tag{3}$$

where $m \in \mathbb{C}$.
This model includes equation of neutrino with $m = 0$ as well as massive spin-$\frac{1}{2}$ particle undergoing a redshifting of its wavelength as the universe expands.

Define the space

$$X(R,k,\ell,m) := \Big\{ \psi(x,t) \in C([0,\infty); H_{(k)}) \Big| \|\psi\|_X := \sup_{t\in[1,\infty)} t^{3\ell/2-|\Im(m)|} \|\psi(t)\|_{H_{(k)}} \le R \Big\}.$$

We assume that the potential $V(x,t) \in \mathscr{B}_{(k;\ell)}$ that implies that all entries of the matrix V belong to the space

$$\mathscr{B}^{(\ell,k)} := \Big\{ v \in C^{\ell,k}_{t,x}([0,\infty)\times\mathbb{R}^3) \Big| \partial_t^j \partial_x^\alpha v(x,t) \in L^\infty([0,\infty)\times\mathbb{R}^3), \forall \alpha,\ |\alpha| \le k,\ \forall j \le \ell \Big\}.$$

We set for simplicity $a_0 = 1$, that is, $a(t) = t^{\ell}$. Below $\alpha^j = \gamma^0\gamma^j$.

Theorem 1. *Let $F = F(\psi) \in C^3(\mathbb{C}^4;\mathbb{C}^4)$ be the Lipschitz continuous with exponent $\alpha > 0$ in the space $H_{(s)}(\mathbb{R}^3)$, $s \geq 3$, function and the potential $V \in \mathscr{B}^{(0,s)}$ is self-adjoint, $V(x,t) = V^*(x,t)$. The problem*

$$\begin{cases} \left(\partial_0 + t^{-\ell}\sum_{j=1,2,3}\alpha^j\partial_j + \frac{3\ell}{2}t^{-1}\mathbb{I}_4 + imt^{-1}\gamma^0 - iV(x,t)\right)\psi = F(\psi)\,, \\ \psi(x,1) = \psi_0(x)\,, \end{cases} \tag{4}$$

with

$$3\ell > 2|\Im(m)| \quad \text{and} \quad \alpha > 2\left(3\ell - 2|\Im(m)|\right)^{-1} \tag{5}$$

for sufficiently small ε and $\psi_0 \in H_{(s)}$, $s > 3/2$, $\|\psi_0\|_{(s)} \leq \varepsilon$, **has a global solution** $\psi \in X(2\varepsilon, s, \ell, m)$.

Condition ($\mathcal{L}$) is

$$\|F(\psi_1) - F(\psi_2)\|_{H_{(s)}(\mathbb{R}^3)}$$
$$\leq C\|\psi_1 - \psi_2\|_{H_{(s)}(\mathbb{R}^3)} \left(\|\psi_1\|^{\alpha}_{H_{(s)}(\mathbb{R}^3)} + \|\psi_2\|^{\alpha}_{H_{(s)}(\mathbb{R}^3)}\right) .$$

The condition on α

$$\alpha > 2\left(3\ell - 2|\Im(m)|\right)^{-1}$$

For the real-valued mass, $m \in \mathbb{R}$, this condition is $\alpha > 2/(3\ell)$.

$\alpha > 1$ if $\ell = 2/3$ and $\alpha > \frac{4}{3}$ if $\ell = 1/2$.

For the Klein-Gordon equation with $m = 0$ for $\ell = 2/3$ if the semilinear term is positive and the Lipschitz continuous with exponent $\alpha < 2$, then, the blowup occurs. (G-Yagdjian'2020). The same is true, if $\ell = 1/2$ and $m = 0$ if $\alpha \leq 4/3$.

For general ℓ blowup occurs if $\alpha < \frac{2}{3-3\ell}$ or $\alpha < \frac{4-\ell}{4-3\ell}$.

We prove the Theorem 1 by the energy estimate and fixed point argument. The metric tensor (1) allows us to develop the energy estimate that is the main tool to prove global existence of small amplitude solutions.

Energy Estimate. Assume that $a(t) = a_0 t^\ell$ and $f \in C([0,\infty); (H_{(k)}(\mathbb{R}^3))^4)$, the potential $V \in \mathscr{B}^{(0,k)}$ is self-adjoint, $V(x,t) = V^*(x,t)$. Then, for the solution of

$$\left(\partial_0 + \frac{1}{a(t)} \sum_{j=1,2,3} \alpha^j \partial_j + \frac{3\dot{a}(t)}{2a(t)} \mathbb{I}_4 + imt^{-1}\gamma^0 - iV(x,t) \right) \psi = f(x,t)\,,$$

in the Sobolev space $H_{(k)}(\mathbb{R}^3)$ one has

$$\begin{aligned} & \|\psi(t)\|_k \\ \le \quad & ct^{-3\ell/2+|\Im(m)|}s^{3\ell/2-|\Im(m)|}\|\psi(s)\|_k \\ & +ct^{-3\ell/2+|\Im(m)|}\int_s^t \tau^{3\ell/2-|\Im(m)|}\|f(\tau)\|_k \, d\tau\,, \quad 0<s\le t\,, \end{aligned}$$

$$\begin{aligned} & \|\psi(t)\|_k \\ \le \quad & ct^{-3\ell/2-|\Im(m)|}s^{3\ell/2+|\Im(m)|}\|\psi(s)\|_k \\ & +ct^{-3\ell/2-|\Im(m)|}\int_t^s \tau^{3\ell/2+|\Im(m)|}\|f(\tau)\|_k \, d\tau\,, \quad 0<t\le s. \end{aligned}$$

Denote by $S(t,s)$ the propagator (fundamental solution for the Cauchy problem), that is, an operator-valued solution of the problem

$$\begin{cases} \left(\partial_0 + t^{-\ell} \sum_{\ell=1,2,3} \alpha^{\ell}\partial_{\ell} + \frac{3\ell}{2} t^{-1}\mathbb{I}_4 + imt^{-1}\gamma^0 - iV(x,t)\right) S(t,s) = 0\,, \\ S(s,s) = I \text{ (identity operator)}\,. \end{cases}$$

Then the solution of the problem $t \geq s > 0$

$$\begin{cases} \left(\partial_0 + t^{-\ell} \sum_{\ell=1,2,3} \alpha^\ell \partial_\ell + \frac{3\ell}{2} t^{-1} \mathbb{I}_4 + imt^{-1}\gamma^0 - iV(x,t) \right) \psi = f(x,t)\,, \\ \psi(x,s) = \psi_0(x)\,, \end{cases}$$

is given by Duhamel's principle

$$\psi(x,t) = S(t,s)\psi_0(x) + \int_s^t S(t,\tau) f(x,\tau)\, d\tau\,.$$

We define the operator $\mathcal{S}$ by

$$\mathcal{S}\psi(x,t) := S(t,1)\psi_0(x) + \int_1^t S(t,\tau)F(\psi(x,\tau))\, d\tau \,.$$

Using obtained energy estimate we prove $\mathcal{S}$ is a contraction

$$\mathcal{S} : X(R,k,\ell,m) \longrightarrow X(R,k,\ell,m)$$

for sufficiently small R and the operator $\mathcal{S}$ maps $X(R,k,\ell,m)$ into itself if $\psi_0 \in H_{(k)}$, $\|\psi_0\|_{H_{(k)}} < \varepsilon$, and ε and R are sufficiently small, namely, if

$$\varepsilon + \left(\frac{1}{\alpha(\frac{3}{2}\ell - |\Im(m)|) - 1} \right) R^{1+\alpha} < R.$$

The Banach fixed-point theorem completes proof of Theorem 1.

The equation

$$\left(\partial_0 + t^{-\ell} \sum_{j=1,2,3} \alpha^j \partial_j + \frac{3\ell}{2} t^{-1} \mathbb{I}_4 + imt^{-1}\gamma^0 - iV(x,t) \right) \psi = F(\psi) ,$$

is a symmetric hyperbolic system, and the local existence of the solution is known. Thus, the local Cauchy problem for this equation is well posed in $C^0([1, T]; (H_s(\mathbb{R}^3))^4)$, $s \geq 3$, for some $T > 1$.

The next theorem states that some local solutions of the large data can be continued to the global ones.

Theorem 2. *Let $m \in \mathbb{R}$, $V \in \mathscr{B}^{(\infty,\infty)}$ is self-adjoint, $V^*(x,t) = V(x,t)$, and $V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0$. Suppose that $a(t) = a_0 t^\ell$ has $\ell > 1/3$ and $F = F(\xi,\eta)$, $F \in C^\infty$, has the form*

$$F\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right) = \alpha\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right) I + i\beta\left(\psi^*\gamma^0\psi\,,\,\psi^*\gamma^0\gamma^5\psi\right)\gamma^5,$$

where α and β are real-valued functions, with

$$\alpha(\xi,\eta) = O(|\xi|+|\eta|), \quad \beta(\xi,\eta) = O(|\xi|+|\eta|), \quad |\xi|+|\eta| \to 0\,.$$

Assume also that $\Psi_0 = \Psi_0(x) \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ satisfies the Lochak-Majorana condition

$$\rho^2(\Psi_0(x)) := |\Psi_0^*(x)\gamma^0\Psi_0(x)|^2 + |\Psi_0^*(x)\gamma^0\gamma^5\Psi_0(x)|^2 = 0 \quad \forall x \in \mathbb{R}^3\,.$$

Then for $\chi_0 \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ *there is* $\varepsilon_0 > 0$ *such that the Cauchy problem*

$$\begin{cases} \left(\mathscr{D}_{FLRW}(t,\partial_t,\partial_x) + \gamma^0 V(x,t)\right)\psi = F\left(\psi^*\gamma^0\psi\,,\, \psi^*\gamma^0\gamma^5\psi\right)\psi, t > 1, \\ \psi(x,1) = \Psi_0(x) + \varepsilon\chi_0(x), \end{cases}$$

with $0 < \varepsilon < \varepsilon_0$ *has a unique solution* $\psi = \psi(x,t)$ *such that* $\psi(t) \in C^\infty(\mathbb{R}^3;\mathbb{C}^4)$ *for all* $t \in (0,\infty)$.

The solution scatters to free solution as $t \to +\infty$.

The proof of the existence of global solution in Theorem 2 is done by the successive approximations.

- We follow approach developed by Bachelot'1989 that appeals to the estimates obtained for the for nonlinear massless Dirac equation in Minkowski spacetime.
- We replace the generators of the Poincaré group with the Fermi operators.
- Obtained energy estimates allow us to expand the result obtained by Bachelot to the Dirac equation in the FLRW spacetime.

Lemma 1. Consider the Dirac equation with the matrix-valued potential function

$$A(x,t) = \alpha(x,t)\mathbb{I}_4 + i\beta(x,t)\gamma^5, \tag{6}$$

where $\alpha, \beta \in C^0([0,\infty); \; L^2(\mathbb{R}^3))$ are real-valued functions. For the solution $\psi \in C^1([1,\infty); L^2(\mathbb{R}^3)) \cap C^0([1,\infty); H_{(1)}(\mathbb{R}^3))$ of the Dirac equation

$$\left(i\gamma^0\partial_0 + it^{-\ell} \sum_{j=1,2,3} \gamma^j \partial_{x_j} + i\frac{3\ell}{2}t^{-1}\gamma^0 - mt^{-1}\mathbb{I}_4 + \gamma^0 V(x,t)\right)\psi = -A\psi\,, \tag{7}$$

the following energy identity holds

$$\begin{aligned}\|\psi(\cdot,t)\|^2_{L^2(\mathbb{R}^3)} &= t^{-3\ell}\|\psi(\cdot,1)\|^2_{L^2(\mathbb{R}^3)} \\ &+2\Im(m)t^{-3\ell}\int_1^t s^{3\ell-1}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi(x,s)\,dx\,ds \\ &-2t^{-3\ell}\int_1^t s^{3\ell}\int_{\mathbb{R}^3}\psi^*(x,s)\Im(V(x,s))\psi(x,s)\,dx\,ds\end{aligned} \tag{8}$$

for all $\quad t\geq 1.$

Lemma 2. Assume that

$$V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0\,. \tag{9}$$

For the solution $\psi \in C^1([0,\infty); L^2(\mathbb{R}^3)) \cap C^0([0,\infty); H_{(1)}(\mathbb{R}^3))$ of the Dirac equation (7) we have

$$\int_{\mathbb{R}^3} \psi^T(x,t)\gamma^2\psi(x,t)\,dx = t^{-3\ell}\int_{\mathbb{R}^3} \psi^T(x,1)\gamma^2\psi(x,1)\,dx\,.$$

In particular, for $z \in \mathbb{C}$, $|z| = 1$,

$$\int_{\mathbb{R}^3} 2\Re\Big(\bar{z}\psi^T(x,t)\gamma^2\psi(x,t)\Big)\,dx = t^{-3\ell}\int_{\mathbb{R}^3} 2\Re\Big(\bar{z}\psi^T(x,1)\gamma^2\psi(x,1)\Big)\,dx\,.$$

Lemma 3. Assume that

$$V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0\,.$$

For the solution

$$\psi \in C^1([1,\infty); L^2(\mathbb{R}^3)) \cap C^0([1,\infty); H_{(1)}(\mathbb{R}^3))$$

of the Dirac equation (7) one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\psi(x,t) - z\gamma^2\overline{\psi(x,t)}|^2\,dx \\
= \quad & t^{-3\ell}\left(\int_{\mathbb{R}^3} |\psi(x,1) - z\gamma^2\overline{\psi(x,1)}|^2\,dx\right) \\
&+4\Im(m)t^{-3\ell}\int_1^t s^{3\ell-1}\int_{\mathbb{R}^3} \psi^*(x,s)\gamma^0\psi(x,s)\,dx\,ds\,.
\end{aligned}$$

Corollary. Assume that

$$V^T(x,t)\gamma^2 + \gamma^2 V(x,t) = 0\,.$$

(i) If $\Im(m) = 0$, then

$$\int_{\mathbb{R}^3} |\psi(x,t) - z\gamma^2\overline{\psi(x,t)}|^2\,dx = t^{-3\ell}\left(\int_{\mathbb{R}^3} |\psi(x,1) - z\gamma^2\overline{\psi(x,1)}|^2\,dx\right)\,.$$

(ii) If $\psi(x,1) - z\gamma^2\overline{\psi(x,1)} = 0$ and $\Im(m) \neq 0$, then

$$\int_{\mathbb{R}^3} |\psi(x,t) - z\gamma^2\overline{\psi(x,t)}|^2 dx = 4\Im(m)t^{-3\ell}\int_1^t s^{3\ell-1}\int_{\mathbb{R}^3}\psi^*(x,s)\gamma^0\psi(x,s)dxds.$$

(iii) if $\psi(x,1) - z\gamma^2\overline{\psi(x,1)} = 0$, then

$$\int_{\mathbb{R}^3} |\psi(x,t) - z\gamma^2\overline{\psi(x,t)}|^2 dx \quad \leq \quad 4|\Im(m)| t^{-3\ell} \int_1^t s^{3\ell-1} \int_{\mathbb{R}^3} \rho(x,s)\, dx ds.$$

The last statement of the corollary contains Lochak-Majorana condition of Theorem 2 written as follows:

$$\rho^2(\psi) = (|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2)^2 + (2\Im(\psi_1\overline{\psi_3}) + 2\Im(\psi_2\overline{\psi_4}))^2 .$$

If $F(\psi^*\gamma^0\psi, \psi^*\gamma^0\gamma^5\psi)\psi$ is the Lipschitz continuous function with exponent $\alpha_L > 0$ in $H_{(6)}(\mathbb{R}^3)$, where $3\ell\alpha_L > 2$, then for every solution $\psi = \psi(x,t)$ of the problem

$$\begin{cases} \left(\mathscr{D}_{FLRW}(t,\partial_t,\partial_x) + \gamma^0 V(x,t)\right)\psi = F\left(\psi^*\gamma^0\psi\,,\ \psi^*\gamma^0\gamma^5\psi\right)\psi, t > 1, \\ \psi(x,1) = \Psi_0(x) + \varepsilon\chi_0(x), \end{cases}$$

there exists the solution $\widetilde{\psi}(x,t)$ of the Cauchy problem for the free Dirac equation

$$\begin{cases} \left(\mathscr{D}_{FLRW}(t,\partial_t,\partial_x) + \gamma^0 V(x,t)\right)\widetilde{\psi} = 0\,, \\ \widetilde{\psi}(x,0) = \widetilde{\psi}(x)\,, \end{cases}$$

such that

$$\lim_{t\to+\infty}\left(\left\|\psi(x,t) - \widetilde{\psi}(x,t)\right\|_{(H_6(\mathbb{R}^3))^4} + \left\|\partial_t\psi(x,t) - \partial_t\widetilde{\psi}(x,t)\right\|_{(H_6(\mathbb{R}^3))^4}\right) = 0$$

Consider the energy integral

$$E(t) = \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx.$$

Theorem 3. *Consider the Cauchy problem*

$$\begin{cases} \Big(i\gamma^0\partial_0 + it^{-\ell}\sum_{j=1,2,3}\gamma^j\partial_{x_j} + i\frac{3\ell}{2}t^{-1}\gamma^0 - mt^{-1}\mathbb{I}_4 + \gamma^0 V(x,t)\Big)\psi = G(\psi)i\gamma^0\psi \\ \psi(x,1) = \psi_0(x) \end{cases} \tag{10}$$

with $\psi_0(x)$ *such that* $\operatorname{supp}\psi_0(x) \subseteq \{x \in \mathbb{R}^3 \mid |x| \le R\}$. *Assume*

$$G(\zeta) = O(|\zeta|),\ \gamma^0 G(\zeta) = G(\zeta)\gamma^0,\ \Re(G(\zeta)\zeta, \overline{\zeta}) \ge c_0|\zeta|^{2+\alpha},\ \alpha > 0\,.$$

Then for **arbitrary size initial data there is no global solution** ψ *of the problem (10) that obeys the finite propagation speed property*

$$\operatorname{supp}\psi(x,t) \subseteq \left\{ x \in \mathbb{R}^3 \,\middle|\, |x| \le R + \mathcal{A}(t) - \mathcal{A}(1) \right\},$$

$$\mathcal{A}(t) := \int_1^t \frac{1}{a(\tau)}\, d\tau\,,$$

in the following cases:

- If $0 < \ell < 1$ and $1 \ge \frac{3\alpha}{2} + \alpha|\Im(m)|$,
- If $1 < \ell$ and $1 \ge \frac{3\alpha}{2}\ell + \alpha|\Im(m)|$,
- If $\ell = 1$ and $1 > \frac{3\alpha}{2} + \alpha|\Im(m)|$,
- If $\ell = 1$ and $1 = \frac{3\alpha}{2} + \alpha|\Im(m)|$. If $\alpha < 2/3$.

Furthermore, in the following cases **for the large initial data there is no global solution of (10)** *that obeys the finite propagation speed property:*

- If $0 < \ell < 1$ and $1 < \frac{3\alpha}{2} + \alpha|\Im(m)|$.
- If $1 < \ell$ and $1 < \frac{3\alpha}{2}\ell + \alpha|\Im(m)|$.
- If $\ell = 1$ and $1 < \frac{3\alpha}{2} + \alpha|\Im(m)|$.
- If $\ell = 1$ and $1 = \frac{3\alpha}{2} + \alpha|\Im(m)|$. If $\alpha \geq 2/3$.

The solution blows up no later than time T_{bu} *depending on* c_0, α, *m*, ψ_0, *and* ℓ *such that*

$$(E(1))^{-\frac{\alpha}{2}} = e^{A(1)\frac{\alpha}{2}} \frac{\alpha}{2} c_0 \int_1^{T_{bu}} (R + \mathcal{A}(t) - \mathcal{A}(1))^{-\frac{3\alpha}{2}} e^{-A(s)\frac{\alpha}{2}} ds$$

and

$$\lim_{t \nearrow T_{bu}} \int_{\mathbb{R}^3} |\psi(x,t)|^2 \, dx = \infty \,,$$

provided that $E(1)$ is sufficiently large.

Thank you very much for your attention!