A relative trace formula in electromagnetic scattering

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Outline of the talk

- Lipschitz domains and Maxwell's equations
- Introduction to some unusual Sobolev spaces
- Layer potential theory for Maxwell's equations
- Where do we fit in?
- Characterisation of Maxwell resolvents in terms of layer potentials
- Statement of the main theorems
- Idea of the proof

Joint with Alexander Strohmaier (Leeds)

Setup

- We assume that $\Omega \subset \mathbb{R}^3$ be an open and bounded (strongly) Lipschitz domain in $\mathbb{R}^3$
- The finitely many connected components will be denoted by Ω_j with some index j, which ranges from 1 to N.
- $\overline{\Omega}$ is a collection of disjoint compact obstacles $\overline{\Omega}_j$ placed in $\mathbb{R}^3$
- $M = \mathbb{R}^3 \backslash \overline{\Omega}$ with Lipschitz boundary $\partial\Omega$. M is connected.
- $X = \mathbb{R}^3 \setminus \partial\Omega = M \cup \Omega$.

Since Ω is a Lipschitz domain we have an almost everywhere defined exterior unit vector field $\nu \in L^\infty(\partial\Omega, \mathbb{R}^3)$.

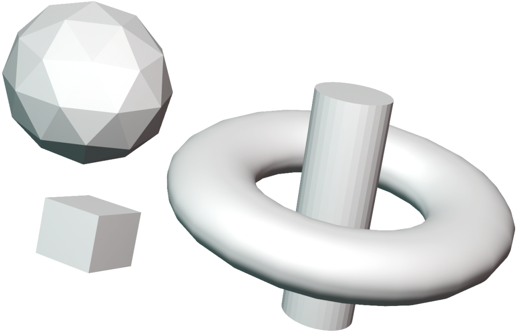

Figure: A Lipschitz domain Ω consisting of four connected components $\Omega_1, \Omega_2, \Omega_3, \Omega_4$.

Maxwell's equations, time harmonic form

We denote by $E = (E_1, E_2, E_3)$, $H = (H_1, H_2, H_3)$ the electric and magnetic fields solving in M or Ω

$$\begin{aligned}
&\operatorname{curl} E - \mathrm{i}\lambda H = 0\\
&\operatorname{div} E = 0\\
&\operatorname{curl} H + \mathrm{i}\lambda E = 0\\
&\operatorname{div} H = 0\\
&\nu \times E = f \quad \text{on} \quad \partial\Omega\\
&\langle \nu, H \rangle = W \quad \text{on} \quad \partial\Omega
\end{aligned}$$

$\Rightarrow$ Results in $-\Delta E - \lambda^2 E = 0$, $\operatorname{div} E = 0$.

Spaces for Maxwell

- $H(\mathrm{curl}, M) = \{f \in L^2(M, \mathbb{C}^3) \mid \mathrm{curl} f \in L^2(M, \mathbb{C}^3)\}$.
- $H(\mathrm{div}, M) = \{f \in L^2(M, \mathbb{C}^3) \mid \mathrm{div} f \in L^2(M)\}$.
- $L^2_{\mathrm{tan}}(\partial\Omega) = \{f \in L^2(\partial\Omega, \mathbb{C}^3) \mid \nu \cdot f = 0 \text{ a.e. on } \partial\Omega\}$.
- $H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$, $H^{-\frac{1}{2}}(\mathrm{Curl}, \partial\Omega)$.

Same for interior problem. First space replaces the $H^1(M)$ space. Last spaces replace $H^{\pm\frac{1}{2}}(\partial\Omega)$

Boundary Conditions

We define $L^2_{\tan}(\partial\Omega) = \{f \in L^2(\partial\Omega, \mathbb{C}^3) \mid \nu \cdot f = 0 \text{ a.e. on } \partial\Omega\}$. Surface divergence is defined on $A \in L^2_{\tan}(\partial\Omega)$ as

$$\langle \mathrm{Div} A, f \rangle_{L^2(\partial\Omega)} = -\int_{\partial\Omega} \langle A, \nabla_{tan} f \rangle d\,\sigma,$$

for f Lipschitz continuous. We have the integration by parts formula with

$$\int_{\partial\Omega} f\,\mathrm{Div} A\, d\sigma = -\int_{\partial\Omega} \langle \nu \times A, (\nu \times \nabla) f \rangle d\,\sigma.$$

So when $E = A$, then $\mathrm{Div} E = 0$ gives $\nu \times E = 0$ **relative boundary conditions**

Lipschitz Problems

In case $\partial\Omega$ is smooth we have

$$H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega) = \{f \in H^{-\frac{1}{2}}(\partial\Omega; T\partial\Omega) \mid \mathrm{Div} f \in H^{-\frac{1}{2}}(\partial\Omega)\},$$
$$H^{-\frac{1}{2}}(\mathrm{Curl}, \partial\Omega) = \{f \in H^{-\frac{1}{2}}(\partial\Omega; T\partial\Omega) \mid \mathrm{Curl} f \in H^{-\frac{1}{2}}(\partial\Omega)\},$$

where Div is the surface divergence on $\partial\Omega$, and Curl is the surface curl. On a general Lipschitz domain this can be defined via Lipschitz coordinate charts, thus locally reducing it to the smooth case. The spaces $H^s_{\mathrm{loc}}(\mathbb{R}^d)$ are invariant under bi-Lipschitz maps if $|s| \leq 1$.

On $H(\mathrm{curl}, M)$ there are two distinguished and well-defined continuous trace maps

$$\gamma_{T,-} : H(\mathrm{curl}, M) \to H^{-\frac{1}{2}}(\mathrm{Curl}, \partial\Omega),$$
$$\gamma_{t,-} : H(\mathrm{curl}, M) \to H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega),$$

which continuously extend the maps $f \mapsto (\nu \times f|_{\partial\Omega}) \times \nu$ and $f \mapsto (\nu \times f|_{\partial\Omega})$ respectively, defined on $C_0(\overline{M}, \mathbb{C}^3)$. Note that for $x \in \partial\Omega$ such that ν_x is defined the map $v \mapsto (\nu_x \times v) \times \nu_x$ is the orthogonal projection onto the tangent space of $\partial\Omega$ at x. Similarly we have the map

$$\gamma_{\nu,-} : H(\mathrm{div}, M) \to H^{-\frac{1}{2}}(\partial\Omega),$$

continuously extending the normal restriction map $f \mapsto \nu \cdot f|_{\partial\Omega}$.

Layer potentials for the Helmoltz Equation

The distributional kernel of the resolvent of the operator $(-\Delta_{\text{free}} - \lambda^2)^{-1}$ is called the Green's function and in dimension three given explicitly by

$$G_{\lambda,\text{free}}(x, y) = \frac{1}{4\pi} \frac{e^{i\lambda|x-y|}}{|x-y|}.$$

The single layer potential operator $\tilde{\mathcal{S}}_\lambda : H^{-\frac{1}{2}}(\partial\Omega) \to H^1_{\text{loc}}(\mathbb{R}^3)$ by

$$\tilde{\mathcal{S}}_\lambda = (-\Delta_{\text{free}} - \lambda^2)^{-1}\gamma^*.$$

The single layer operator is defined by taking the trace $\mathcal{S}_\lambda = \gamma_+\tilde{\mathcal{S}}_\lambda = \gamma_+(-\Delta_{\text{free}} - \lambda^2)^{-1}\gamma^*$.

Representations of Helmholtz Solutions

If everything is smooth $u_\lambda(x)$ with

$$u_\lambda(x) = \int_{\partial\Omega} G_{\lambda,\mathrm{free}}(x,y) g(y)\, d\sigma(y) \qquad x \in X$$

and

$$g(y) = \mathcal{S}_\lambda^{-1} h(x)$$

where

$$\mathcal{S}_\lambda f = \int_{\partial\Omega} G_{\lambda,\mathrm{free}}(x,y) f(y)\, d\sigma(y) \qquad x \in \partial\Omega$$

solves the Dirichlet problem.

Maxwell Problems

Solution to $\lambda \neq 0$, $\operatorname{Im} \lambda > 0$ of

$$(-\Delta - \lambda^2)u = 0 \quad \text{in} \quad X$$
$$u|_{\partial\Omega} = h$$

is $u = \tilde{\mathcal{S}}_\lambda \mathcal{S}_\lambda^{-1}(h)$.
The operator $\mathcal{S}_\lambda^{-1}$ coincides with the Dirichlet-to-Neumann operator. *For the function case, and Lipschitz domains, this is well studied, not for Maxwell!*

Problems:

1) There are problems forming the solutions to Maxwell's equations in the case of "zero" boundary conditions.
2) Representation formula do not really exist, $\lambda = 0$ in the Maxwell case.

Boundary Layer Operators for Maxwell

In both cases, if $\lambda \neq 0$ the solution can be written as boundary layer potential of the form

$$E(x) = (\tilde{\mathcal{L}}_\lambda a)(x) = \mathrm{curl}^2 \langle a, G_\lambda(x,\cdot)\rangle_{\partial\Omega}, \quad H(x) = \frac{\mathrm{i}\,\mathrm{curl} E}{-\lambda} \quad x \notin \Omega$$

with the density $a \in H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$, which satisfies $\mathcal{L}_\lambda a = f$.
Basically $\tilde{\mathcal{L}}_\lambda \mathcal{L}_\lambda^{-1} f = E$
$\Rightarrow$ bad things happen at $\lambda = 0$, and on "$\mathrm{Div}0$".

The *Maxwell single layer operator* $\mathcal{L}_\lambda$ is then defined for all $\lambda \in \mathbb{C}$ as a map

$$\mathcal{L}_\lambda : H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega) \to H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega), \quad u \mapsto \gamma_t \tilde{\mathcal{L}}_\lambda$$

and is a holomorphic family of bounded operators on $H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$ in λ.

Where are we in the literature?

Not all but some names....

- Taylor, Mitrea, Mitrea (95,97,03), Mitrea– $L^p \to L^q$ estimates: Hodge Laplacian (early 10's), topology, C^2 domains (TMMVasy)
- Buffa, Hiptmair, Costabel (2002-3, cont.)– numerical analysis: Set up function spaces so the BEM works
- Kirsch and Hettlich– Book (2015) collected facts in a coherent fashion
- Modern- groups at U Bath- Spence, UCL- Betcke
- Gestzy, Simon, Lesch- spectral shift functions on manifolds

Lipschitz domains at $\lambda = 0$ with relative boundary conditions... Challenge Accepted!

Setup

On the boundary one has well-defined anisotropic Sobolev spaces $H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$ and the Maxwell electric field operator

$$\mathcal{L}_\lambda : H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega) \to H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$$

This can be done for each object separately and one can assemble the individual parts $\mathcal{L}_{\lambda,\partial\Omega_j} : H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega_j) \to H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega_j)$ into an operator $\mathcal{L}_{D,\lambda} = \oplus_{j=1}^{N} \mathcal{L}_{\lambda,\partial\Omega_j}$ acting on $H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$.

Theorem

The operator $\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}$ *is well-defined and a trace-class perturbation of the identity for any complex* λ *with* $\mathrm{Im}(\lambda) > 0$*. It therefore has a well-defined Fredholm determinant* $\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1})$ *on the space* $H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$*. Let* δ *be the minimal distance between separate objects. Then for any* $0 < \delta' < \delta$ *the function*

$$\Xi(\lambda) = \log\det(\mathcal{L}_\lambda \mathcal{L}_{D,\lambda}^{-1}),$$

extends to a holomorphic function in a neighborhood of the closed upper half space with

$$|\Xi(\lambda)| \leq C e^{-\mathrm{Im}(\lambda)\delta}$$

for λ *in any sector about the positive imaginary axis of angle strictly less than* π*.*

Assume $0 < \epsilon \leq \pi$ and let $\mathfrak{S}_\epsilon$ be the open sector

$$\mathfrak{S}_\epsilon = \{z \in \mathbb{C} \mid z \neq 0, |\arg(z)| < \epsilon\}$$

The space $\mathcal{E}_\epsilon$ will be defined by

$$\mathcal{E}_\epsilon = \\ \{f : \mathfrak{S}_\epsilon \to \mathbb{C} \mid f \text{ is holomorphic in } \mathfrak{S}_\epsilon, \\ \exists \alpha > 0, \forall \epsilon_0 > 0, \ |f(z)| = O(|z|^\alpha e^{\epsilon_0 |z|})\}.$$

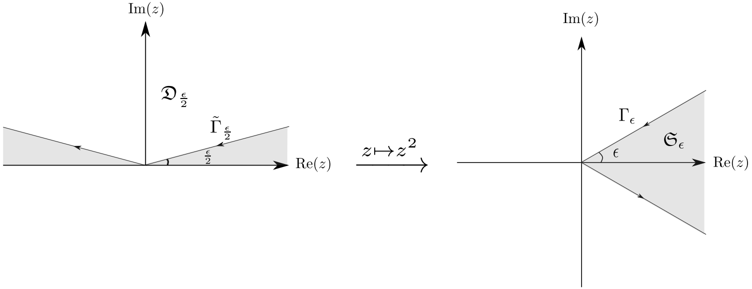

Figure: The sectors $\mathfrak{S}_\epsilon$, $\mathfrak{D}_{\frac{\epsilon}{2}}$ and the corresponding contours.

We define the space $\mathcal{P}_\epsilon$ as the set of functions in $\mathcal{E}_\epsilon$ whose restriction to $[0, \infty)$ is polynomially bounded and that extend continuously to the boundary of $\mathfrak{S}_\epsilon$ in the logarithmic cover of the complex plane. The space $\mathcal{P}_\epsilon$ contains in particular $f(z) = z^\alpha, \quad \alpha > 0$ for any $0 < \epsilon \leq \pi$.

Definition

The space $\widetilde{\mathcal{P}}_\epsilon$ is defined to be the space of functions f such that $f(\lambda) = \lambda^{-2} g(\lambda^2)$ for some $g \in \mathcal{P}_\epsilon$. In particular $f(\lambda) = O(\lambda^a)$ for some $a > -2$ near $\lambda = 0$.

Relative resolvent difference

For $f \in \widetilde{\mathcal{P}}_\epsilon$ we define the *relative operator*

$$D_{\mathrm{rel},f} = f((-\Delta_{\mathrm{rel}})^{\frac{1}{2}})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{\frac{1}{2}})\mathrm{curl}\,\mathrm{curl} - \sum_{j=1}^{N} \left(f((-\Delta_{j,\mathrm{rel}})^{\frac{1}{2}})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{\frac{1}{2}})\mathrm{curl}\,\mathrm{curl} \right),$$

where $f(\lambda) = g(\lambda^2)$.
$B_{\mathrm{rel},f}$ is just

$$B_{\mathrm{rel},f} = f((-\Delta_{\mathrm{rel}})^{\frac{1}{2}})\mathrm{curl}\,\mathrm{curl} - f((-\Delta_{\mathrm{free}})^{\frac{1}{2}})\mathrm{curl}\,\mathrm{curl}$$

Notice that the difference without the $\mathrm{curl}\,\mathrm{curl}$ is not trace class for $f(\lambda) = 1/\lambda$

Theorem

If $f \in \widetilde{\mathcal{P}}_\epsilon$, then operators $D_{\mathrm{rel},f}$ and $D_{\mathrm{abs},f}$ extend to trace-class operators $L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\mathbb{R}^3, \mathbb{C}^3)$ and

$$\mathrm{tr}(D_{\mathrm{rel},f}) = \mathrm{tr}(D_{\mathrm{abs},f}) = \frac{\mathrm{i}}{2\pi} \int_{\tilde{\Gamma}_{\frac{\epsilon}{2}}} \Xi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^2 f(\lambda))\mathrm{d}\lambda,$$

where the contour $\tilde{\Gamma}_{\frac{\epsilon}{2}}$ is the clockwise-oriented boundary of a sector that includes the imaginary axis.

Theorem

We have that B_f *has an integral kernel* $\kappa \in C^\infty(X \times X, \mathrm{Mat}(3, \mathbb{C}))$*, which is smooth away from the boundary. If* $\Omega_0 \subset X$ *has positive distance to the boundary* $\partial\Omega$ *and* p_{Ω_0} *the orthogonal projection* $L^2(\mathbb{R}^3, \mathbb{C}^3) \to L^2(\Omega_0, \mathbb{C}^3)$*, then* $p_{\Omega_0} B_f p_{\Omega_0}$ *extends to a trace-class operator with trace equal to the convergent integral*

$$\int_{\Omega_0} \mathrm{tr}\left(\kappa(x,x)\right) \mathrm{d}x.$$

For large $|x|$ *we have the estimate*

$$\|\kappa(x,x)\| \leq C \frac{1}{|x|^{6+a}},$$

if $f(z) = O(|z|^a)$ *for* $|z| < 1$*.*

Proposition

Assume that $\mathrm{Im}(\lambda) > 0$. *For* $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$ *we have the following formulae for the difference of resolvents:*

$$\left((-\Delta_{\mathrm{rel}} - \lambda^2)^{-1} - (-\Delta_{\mathrm{free}} - \lambda^2)^{-1}\right) \mathrm{curl}\mathrm{curl} f =$$
$$- \tilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \tilde{\mathcal{L}}_\lambda^t f,$$

Here $\tilde{\mathcal{L}}_\lambda^t$ *is the transpose operator to* $\tilde{\mathcal{L}}_\lambda$ *obtained from the real* L^2*-inner product, i.e.* $\tilde{\mathcal{L}}_\lambda^t f = \overline{\tilde{\mathcal{L}}_\lambda^* \overline{f}}$.

We know that $\tilde{\mathcal{L}}_\lambda$ maps to functions satisfying the Helmholtz equation $\left(-\Delta - \lambda^2\right) v = 0$. We need to show that, given $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^3)$, the function

$$u = (-\Delta_{\mathrm{free}} - \lambda^2)^{-1} \mathrm{curl}\,\mathrm{curl} f - \tilde{\mathcal{L}}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \tilde{\mathcal{L}}_\lambda^{t} f$$

satisfies relative boundary conditions. Since clearly $\mathrm{div}\, u = 0$ we only need to check that $\gamma_t u = 0$.

$\gamma_t u = \gamma_t \mathrm{curl}\,\mathrm{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f - \mathcal{L}_\lambda (\mathcal{L}_\lambda)^{-1} (\nu \times) \tilde{\mathcal{L}}_\lambda^{t} f =$
$\gamma_t \mathrm{curl}\,\mathrm{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f - (\nu \times) \gamma_T \mathrm{curl}\,\mathrm{curl}(-\Delta_{\mathrm{free}} - \lambda^2)^{-1} f =$
$0,$

We now define the voltage-to-current mappings $\Lambda_\lambda^\pm$ be defining

$$\Lambda_\lambda^\pm : \gamma_t(E) \to \gamma_t(H) \qquad (\Lambda_\lambda^\pm)^2 = -I$$

where (E, H) are solutions to the interior and exterior boundary value problem for the Maxwell system. The mapping Λ_λ^+ is therefore well defined for any $\lambda \in \mathbb{C}$ which is not a Maxwell eigenvalue or zero. The mapping Λ_λ^- is well defined for all non-zero λ in the closed upper half space.

If $u = \tilde{\mathcal{M}}_\lambda a = \operatorname{curl} \tilde{\mathcal{S}}_\lambda a$ then we have the jump conditions

$$\gamma_{t,\pm} u = \mp \frac{1}{2} a + \mathcal{M}_\lambda a \quad \gamma_{t,\pm} \operatorname{curl} u = \mathcal{L}_\lambda a.$$

Moreover, the operator $\tilde{\mathcal{L}}_\lambda a$ is

$$\tilde{\mathcal{L}}_\lambda a = \nabla \tilde{\mathcal{S}}_\lambda \operatorname{Div} a + \lambda^2 \tilde{\mathcal{S}}_\lambda a \quad a \in H^{-\frac{1}{2}}(\operatorname{Div}, \partial\Omega).$$

We have the following identities

$$\mathcal{L}_\lambda = -\mathrm{i}\lambda\Lambda_\lambda^{\pm}\left(\mp\frac{1}{2} + \mathcal{M}_\lambda\right) = -\mathrm{i}\lambda\left(\pm\frac{1}{2} + \mathcal{M}_\lambda\right)\Lambda_\lambda^{\pm},$$

and as a consequence

$$\mathrm{i}\lambda^{-1}\mathcal{L}_\lambda(\Lambda_\lambda^{+} - \Lambda_\lambda^{-}) = I \quad \text{and} \quad \mathcal{L}_\lambda^2 = \lambda^2\left(-\frac{1}{2} + \mathcal{M}_\lambda\right)\left(\frac{1}{2} + \mathcal{M}_\lambda\right).$$

$\Rightarrow$ chasing all the identities down gives cancellation of the pole at zero.

Theorem

We have

$$i\Lambda_\lambda^+ = \frac{1}{\lambda}T + \lambda U_\lambda,$$

where

$$TA = \sum_{k=1}^{\beta_1} \langle A, \psi_{0,k}\rangle_{L^2(\partial\Omega)} \gamma_t(\psi_{0,k}) + \sum_{\lambda_{N,k}>0} \frac{1}{\lambda_{N,k}^2} \langle A, \nabla v_k\rangle_{L^2(\partial\Omega)} \gamma_t(\nabla v_k),$$

$$U_\lambda A = \sum_{k=1}^{\infty} \frac{1}{\lambda^2 - \mu_k^2} \langle A, \psi_k\rangle_{L^2(\partial\Omega)} \gamma_t(\psi_k)$$

for $A \in H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$*. Both sums converge in* $H^{-\frac{1}{2}}(\mathrm{Div}, \partial\Omega)$*.*
Here $\beta_1 = \dim \mathcal{H}^1_{abs}(\Omega)$ *is the first Betti number of the domain.*
We have $T^2 = 0$ *and* $TU_\lambda + U_\lambda T = I - \lambda^2 U_\lambda$*.*