Huygens' principle for the fields in spacetime of non-constant curvature

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Abstract:

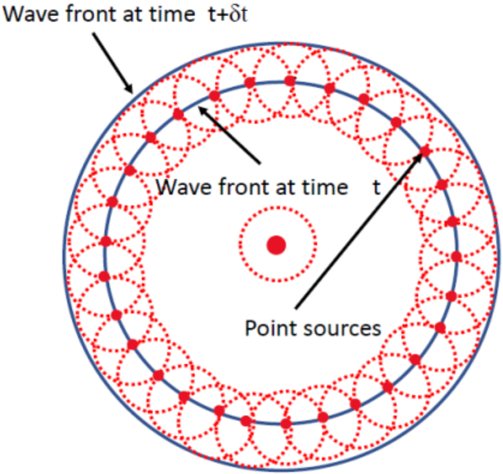
In this talk we will discuss sufficient and necessary conditions for the validity of the Huygens' principle for the Dirac and Klein-Gordon operators in the non-constant curvature spacetimes of the Friedmann-Lemaitre-Robertson-Walker models of cosmology. The Huygens' principle will be considered for the equations of scalar and spinor massless and massive fields undergoing a red shift of their wavelengths as the universe expands.

See arXiv:2112.07808v2

Huygens' principle

- Huygens in his *Traité de la Lumière*, published in 1690, discussed the process of the propagation of light by the aid of a new principle, which has since been generally known by his name.
- At that time light was regarded as a disturbance in a medium, the aether, of much the same general character as sound in air. Huygens supposed that, at any instant $t = t_0$, a point-source of light generates a disturbance which is propagated into the surrounding medium as an isolated spherical wave which expands with a large constant velocity, the velocity of light. This initial disturbance develops into the disturbance at the subsequent instant $t = t_1$ through a succession of states at the intermediate instants, and a knowledge of the state at any intermediate instant $t = t'$ suffices to determine the state at the instant $t = t_1$.
- Thus, if we regard each element of the isolated wave at the instant $t = t'$ as the center of a new disturbance, the actual effect at the instant $t = t_1$ is the resultant of all these secondary effects, the actual wave the envelope of the secondary waves.

Huygens' principle



Huygens' principle, J. Hadamard

J. Hadamard in 1923 analyzed the principle in the form of a syllogism.

- A. (Major premiss.) In order to determine the effect at the instant $t = t_1$ of a luminous phenomenon caused by a given disturbance at the initial instant $t = t_0$, we may calculate the state at some intermediate instant $t = t'$ and from that deduce the state at the instant $t = t_1$.
- B. (Minor premiss.) If at the instant $t = t_0$, or, more precisely, in the short interval $t_0 - \varepsilon \leq t \leq t_0$, we produce a luminous disturbance localized in the immediate neighborhood of a point O, the effect at the subsequent instant $t = t'$ is localized in a very thin spherical shell with center O and radius $c(t' - t_0)$, where c is the velocity of light.
- C. (Conclusion.) In order to calculate the effect at the instant $t = t_1$ due to a luminous disturbance localized at O at the instant $t = t_0$, we may replace the initial disturbance by a suitable system of luminous disturbances taking place at the intermediate instant $t = t'$ and distributed over the surface of the sphere with center O and radius $c(t' - t_0)$.
- Analyzed in this way, Huygens' principle is seen to involve three propositions, and different authors have attached the name "Huygens' principle" indiscriminately to any one of the three.

Huygens' principle. Kirchoff's formula

- *Kirchoff's formula* formula for the Cauchy problem for three-dimensional wave equation

$$\begin{cases} u_{tt} - (u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}) = 0 & x \in \mathbb{R}^3, \; t \in [0, \infty) \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^3 \end{cases}$$

  is given by $(S_t(x_1, x_2, x_3) := \{\alpha \in \mathbb{R}^3 \,|\, |\alpha - x| = t\})$

$$\begin{aligned} u(x_1, x_2, x_3, t) &= \frac{\partial}{\partial t} \frac{1}{4\pi} \iint_{S_t(x_1,x_2,x_3)} \frac{g(\alpha_1, \alpha_2, \alpha_3)}{t} \, d\sigma_t \\ &\quad + \frac{1}{4\pi} \iint_{S_t(x_1,x_2,x_3)} \frac{h(\alpha_1, \alpha_2, \alpha_3)}{t} \, d\sigma_t \,. \end{aligned}$$

- We use the definition of the Huygens principle due to Hadamard as the absence of tails. Thus, the field equations satisfy the Huygens principle if and only if the solution has no tail, that is, solution depends on the source distributions on the past null cone of the field only and not on the sources inside the cone.

Huygens' principle (no tail)

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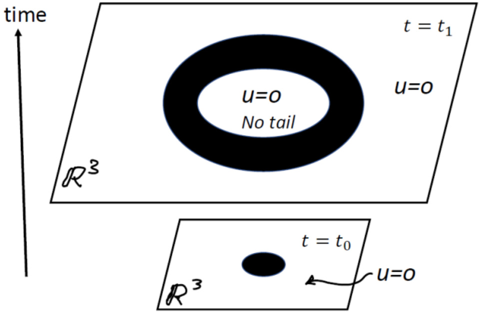


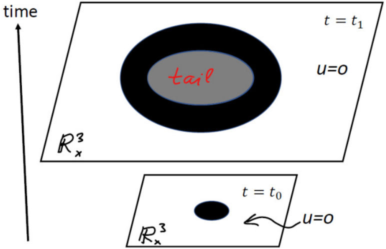
Huygens' principle (tail)

$n = 2$. For the Cauchy Problem

$$\begin{cases} u_{tt} - (u_{x_1x_1} + u_{x_2x_2}) = 0 & x \in \mathbb{R}^2,\ t \in [0, \infty) \\ u(x_1,x_2,0) = g(x_1,x_2), \quad u_t(x_1,x_2,0) = h(x_1,x_2), & x = (x_1,x_2) \in \mathbb{R}^2 . \end{cases}$$

Poisson's formula

$$\begin{aligned} u(x_1,x_2,t) &= \frac{\partial}{\partial t}\frac{1}{2\pi}\iint_{\alpha_1^2+\alpha_2^2\le t^2} \frac{g(x_1+\alpha_1, x_2+\alpha_2)}{\sqrt{t^2-\alpha_1^2-\alpha_2^2}}\, d\alpha_1\, d\alpha_2 \\ &\quad + \frac{1}{2\pi}\iint_{\alpha_1^2+\alpha_2^2\le t^2} \frac{h(x_1+\alpha_1, x_2+\alpha_2)}{\sqrt{t^2-\alpha_1^2-\alpha_2^2}}\, d\alpha_1\, d\alpha_2 . \end{aligned}$$



$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \quad \mu, \nu = 0, 1, 2, 3, \tag{1}$$

The scale factor $a(t) = a_0 t^\ell$, $\ell \in \mathbb{R}$, $a_0 \neq 0$, $t > 0$, and $x \in \mathbb{R}^3$, $x_0 = t$.

- If $\ell = 0$ Minkowski spacetime
- If $\ell < 0$ the spacetime is contracting.
- In the case of $\ell > 1$ the expansion is accelerating (with horizon),
- for $0 < \ell < 1$ the expansion is decelerating.
- If $\ell = 1$ the Milne spacetime.
- If $a(t) = a_0 t^{2/3}$ Einstein-de Sitter (matter dominated) universe
- If $a(t) = a_0 t^{1/2}$ the radiation dominated universe

FLRW spacetime of non-constant curvature 2

The Einstein's summation convention is used.

- Christoffel symbols

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

- Riemann tensor

$$R^i{}_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{kn} - \Gamma^i_{nm}\Gamma^n_{kl}$$

- Ricci tensor $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$
- Scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$
- G is the Newtonian gravitational constant
- $T_{\mu\nu}$ is the energy-momentum tensor
- Λ is a cosmological constant.

The Einstein equations for the metric (tensor) $g_{\mu\nu} = g_{\mu\nu}(x_0, x_1, x_2, x_3)$ are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

Christoffel symbols

$$\begin{aligned}
\Gamma^1_{2\,2} &= \ell t^{2\ell-1}\\
\Gamma^1_{3\,3} &= \ell t^{2\ell-1}\\
\Gamma^1_{4\,4} &= \ell t^{2\ell-1}\\
\Gamma^2_{2\,1} &= \frac{\ell}{t}\\
\Gamma^3_{3\,1} &= \frac{\ell}{t}\\
\Gamma^4_{4\,1} &= \frac{\ell}{t}
\end{aligned},$$

Riemann tensor

$$\begin{aligned}
R^1_{2\,2\,1} &= (\ell-1)\ell\left(-t^{2\ell-2}\right)\\
R^1_{3\,3\,1} &= (\ell-1)\ell\left(-t^{2\ell-2}\right)\\
R^1_{4\,4\,1} &= (\ell-1)\ell\left(-t^{2\ell-2}\right)\\
R^2_{1\,2\,1} &= -\frac{(\ell-1)\ell}{t^2}\\
R^2_{3\,3\,2} &= \ell^2\left(-t^{2\ell-2}\right)\\
R^2_{4\,4\,2} &= \ell^2\left(-t^{2\ell-2}\right)\\
R^3_{1\,3\,1} &= -\frac{(\ell-1)\ell}{t^2}\\
R^3_{2\,3\,2} &= \ell^2 t^{2\ell-2}\\
R^3_{4\,4\,3} &= \ell^2\left(-t^{2\ell-2}\right)\\
R^4_{1\,4\,1} &= -\frac{(\ell-1)\ell}{t^2}\\
R^4_{2\,4\,2} &= \ell^2 t^{2\ell-2}\\
R^4_{3\,4\,3} &= \ell^2 t^{2\ell-2}
\end{aligned},$$

Ricci tensor

$$\begin{aligned}
R_{1\,1} &= -\frac{3(\ell-1)\ell}{t^2}\\
R_{2\,2} &= \ell(3\ell-1)t^{2\ell-2}\\
R_{3\,3} &= \ell(3\ell-1)t^{2\ell-2}\\
R_{4\,4} &= \ell(3\ell-1)t^{2\ell-2}
\end{aligned},$$

Scalar curvature

$$R = \frac{6\ell(2\ell-1)}{t^2}$$

Einstein Tensor

$$\begin{aligned}
G_{1\,1} &= \frac{3\ell^2}{t^2}\\
G_{2\,2} &= \ell(3\ell-2)\left(-t^{2\ell-2}\right)\\
G_{3\,3} &= \ell(3\ell-2)\left(-t^{2\ell-2}\right)\\
G_{4\,4} &= \ell(3\ell-2)\left(-t^{2\ell-2}\right)
\end{aligned}$$

Christoffel symbols $\Gamma^1_{2\,2} = \ell t^{2\ell-1}$

Riemann tensor $R^1_{2\,2\,1} = (\ell-1)\ell\left(-t^{2\ell-2}\right)$

Ricci tensor $R_{1\,1} = -\dfrac{3(\ell-1)\ell}{t^2}$

Scalar curvature $R = \dfrac{6\ell(2\ell-1)}{t^2}$

Einstein Tensor $G_{2\,2} = \ell(3\ell-2)\left(-t^{2\ell-2}\right)$

- If $\ell = 0$ Minkowski spacetime
- If $\ell = 1$ the Milne spacetime.
- If $\ell = 2/3$ Einstein-de Sitter (matter dominated) universe
- If $\ell = 1/2$ the radiation dominated universe

Volkmar Wünsch's theorem

It was proved by Wünsch'85 that if the massive ($m \neq 0$) Dirac equation

$$i\gamma^0\partial_t\Psi + i\frac{1}{a(t)}\gamma^1\partial_{x_1}\Psi + i\frac{1}{a(t)}\gamma^2\partial_{x_2}\Psi + i\frac{1}{a(t)}\gamma^3\partial_{x_3}\Psi + i\frac{3\dot{a}(t)}{2a(t)}\gamma^0\Psi - m_0\mathbb{I}_4\Psi = 0\,,$$

obeys the Huygens' principle, then spacetime has a constant curvature.

Theorem [Wünsch'85]

The the massive field equation satisfy Huygens' principle if and only if four-dimensional pseudo-Riemannian manifold (M, g) is space-time of constant curvature

$$R = \frac{6}{s}\frac{c^2 m_0^2}{\hbar^2}\,,$$

where s is the spin.

This corresponds just to the de Sitter space-time with $a(t) = e^{Ht}$.

The model Dirac field with red shift in curved spacetime

Accordingly, it was admitted K.Y & A.Galstian [JDE 2021] that the mass of the field (particle) can be changing in time and vanishing at future infinity.
The model is determined by the Dirac operator

$$\mathscr{D}(t,\partial_t,\partial_x) := i\gamma^0\partial_t + i\frac{1}{a(t)}\gamma^1\partial_{x_1} + i\frac{1}{a(t)}\gamma^2\partial_{x_2} + i\frac{1}{a(t)}\gamma^3\partial_{x_3} + i\frac{3\dot{a}(t)}{2a(t)}\gamma^0 - \frac{m}{t}\mathbb{I}_4\,, \tag{2}$$

where $m \in \mathbb{C}$,

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & -\mathbb{I}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} \mathbb{O}_2 & \sigma^k \\ -\sigma^k & \mathbb{O}_2 \end{pmatrix}, \quad k = 1,2,3\,.$$

Here σ^k are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\mathbb{I}_n$, $\mathbb{O}_n$ denote the $n \times n$ identity and zero matrices, respectively.
Thus, the field is undergoing a red shift of its wavelength.

Fundamental solutions for model Dirac field in curved spacetime

This model includes a neutrino with $m = 0$ as well as a massive spin-$\frac{1}{2}$ particle undergoing a red shifting of its wavelength as the universe expands.
Thus, we consider the Dirac equation

$$\mathscr{D}(t, \partial_t, \partial_x)\Psi = F \,. \tag{3}$$

- Recall that a retarded fundamental solution for the Dirac operator is a 4×4 matrix $\mathcal{E}^{ret} = \mathcal{E}^{ret}(x, t; x_0, t_0; m)$ with the operator-valued entries that solves

$$\mathscr{D}(t, \partial_t, \partial_x)\mathcal{E}(x, t; x_0, t_0; m) = \delta(x - x_0)\delta(t - t_0)\mathbb{I}_4, \;\; (x, t), (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+, \tag{4}$$

  and with the support in the causal future $D_+(x_0, t_0)$ of $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+$.
- The advanced fundamental solution $\mathcal{E}^{adv} = \mathcal{E}^{adv}(x, t; x_0, t_0; \, m)$ solves the equation (4) and has the support in the causal past $D_-(x_0, t_0)$.
- The forward and backward light cones are the boundaries of

$$D_\pm(x_0, t_0) := \left\{(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+ \, ; \, |x - x_0| \leq \pm(\phi(t) - \phi(t_0))\right\} ,$$

  where $\phi(t) := \frac{1}{1-\ell} t^{1-\ell}$ if $\ell \neq 1$.

The right co-factor for model Dirac field in curved spacetime

Define for upper and lower 2-spinors, two more γ-matrices:

$$\gamma^U = \begin{pmatrix} \mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{O}_2 \end{pmatrix} = \frac{1}{2}(\mathbb{I}_4 + \gamma^0), \quad \gamma^L = \begin{pmatrix} \mathbb{O}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \mathbb{I}_2 \end{pmatrix} = \frac{1}{2}(\mathbb{I}_4 - \gamma^0)\,,$$

Define the right co-factor

$$\begin{aligned} & \mathscr{D}^{co}(t,\partial_t,\partial_x) \qquad (5)\\ := \quad & it^{-\frac{\ell}{2}}\gamma^0(t^{im}\gamma^U + t^{-im}\gamma^L)\frac{\partial}{\partial t} + it^{-\frac{3\ell}{2}}\sum_{k=1}^{3}\gamma^k(t^{im}\gamma^U + t^{-im}\gamma^L)\frac{\partial}{\partial x_k} \end{aligned}$$

of $\mathscr{D}(t,\partial_t,\partial_x)$ such that the composition $\mathscr{D}(t,\partial_t,\partial_x)\mathscr{D}^{co}(t,\partial_t,\partial_x)$ is a diagonal matrix

$$\mathscr{D}(t,\partial_t,\partial_x)\mathscr{D}^{co}(t,\partial_t,\partial_x) = -\begin{pmatrix} t^{-a}\mathbb{I}_2 P(t,\partial_t,\partial_x;m) & \mathbb{O}_2 \\ \mathbb{O}_2 & t^{-\omega}\mathbb{I}_2 P(t,\partial_t,\partial_x;-m) \end{pmatrix}.$$

where $a := \frac{\ell}{2} - im$, $\omega := \frac{\ell}{2} + im$, and

$$P(t,\partial_t,\partial_x;m) := \partial_t^2 - t^{-2\ell}\Delta + t^{-1}(\ell + 2im)\partial_t\,, \qquad (6)$$

The Structure of Fundamental Solution

Let Δ be the Laplace operator in $\mathbb{R}^3$. Denote by $\mathcal{E}^w(x,t)$ the fundamental solution to the Cauchy problem for the wave equation in the Minkowski spacetime

$$\mathcal{E}^w_{tt} - \Delta \mathcal{E}^w = 0\,, \quad \mathcal{E}^w(x,0) = \delta(x)\,, \quad \mathcal{E}^w_t(x,0) = 0\,.$$

The fundamental solution that is responsible for the Huygens' principle is given by **Theorem 1.2 [JDE 2021]:** for every positive $\varepsilon > 0$ and $t > \varepsilon$ the fundamental solution $\mathcal{E}_+(x,t;x_0;m;\varepsilon)$ to the Cauchy problem, that is,

$$\begin{cases} \mathscr{D}(t,\partial_t,\partial_x)\mathcal{E}_+(x,t;x_0;m;\varepsilon) = \mathbb{O}_4, \\ \mathcal{E}_+(x,\varepsilon;x_0;m;\varepsilon) = \delta(x-x_0)\mathbb{I}_4\,, \end{cases}$$

is

$$\begin{aligned} & \mathcal{E}_+(x,t;x_0;m;\varepsilon) \\ = \quad & -i\varepsilon^{1+\frac{\ell}{2}-im}(1-\ell)^{-1}\mathscr{D}^{co}(x,t,\partial_t,\partial_x)\gamma^0 \\ & \times \int_0^{\phi(t)-\phi(\varepsilon)} \begin{pmatrix} K_1\left(r,t;m;\varepsilon\right)\mathbb{I}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & K_1\left(r,t;-m;\varepsilon\right)\mathbb{I}_2 \end{pmatrix} \mathcal{E}^w(x-x_0,r)\,dr, \end{aligned}$$

The Representation of Solution

$$K_1(r,t;m;\varepsilon) := 2^{2i\frac{m}{1-\ell}}\phi(\varepsilon)^{2i\frac{m}{1-\ell}-1}\left((\phi(t)+\phi(\varepsilon))^2-r^2\right)^{-i\frac{m}{1-\ell}} \times F\left(i\frac{m}{1-\ell}, i\frac{m}{1-\ell}; 1; \frac{(\phi(t)-\phi(\varepsilon))^2-r^2}{(\phi(t)+\phi(\varepsilon))^2-r^2}\right).$$

$F(\alpha,\beta;\gamma;z)$ is the hypergeometric function. We use the operator ($\varphi \in C_0^\infty(\mathbb{R}^n)$)

$$\mathcal{K}_1(x,t,D_x;m;\varepsilon)[\varphi](x,t) := -i\varepsilon^{1+\frac{\ell}{2}-im}(1-\ell)^{-1}\int_0^{\phi(t)-\phi(\varepsilon)} K_1(r,t;m;\varepsilon)\int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr. \tag{7}$$

Theorem 1.3 [JDE 2021]. The solution to the Cauchy problem

$$\begin{cases}\mathscr{D}(t,\partial_t,\partial_x)\Psi(x,t)=0\,, & t>\varepsilon>0\,,\\ \Psi(x,\varepsilon)=\Phi(x)\,,\end{cases} \tag{8}$$

with $m\in\mathbb{C}$, is

$$\Psi(x,t)=\mathscr{D}^{co}(t,\partial_t,\partial_x)\gamma^0\begin{pmatrix}\mathcal{K}_1(x,t,D_x;m;\varepsilon)\mathbb{I}_2 & \mathbb{O}_2\\ \mathbb{O}_2 & \mathcal{K}_1(x,t,D_x;-m;\varepsilon)\mathbb{I}_2\end{pmatrix}[\Phi](x,t), \tag{9}$$

Huygens' principle in spacetime of non-constant curvature

We say that the equation obeys the Huygens' principle if the solution Ψ vanishes at all points which cannot be reached from the support of initial data Φ by a null geodesic.

Theorem [K.Y., ArXiv 2021]

The solution of the Dirac equation (8) with $\ell \in \mathbb{R}$, $\ell \neq 1$, and the mass $m \in \mathbb{C}$ obeys the Huygens' principle if and only if the mass term takes the values $m = 0, \pm i(\ell - 1)$.

The Huygens' principle's property is a local one, that is, it can be verified for the small time.
The proof of the necessity part is based on the large time asymptotics.

Proof of Sufficiency $m = 0$

The fundamental solution $\mathcal{E}^w$ to the Cauchy problem for the wave equation in the Minkowski spacetime can be written as $\mathcal{E}^w = \partial_t \mathscr{V}^w$, where

$$\mathscr{V}^w_{tt} - \Delta \mathscr{V}^w = 0\,, \quad \mathscr{V}^w(x,0) = 0\,, \quad \mathscr{V}^w_t(x,0) = \delta(x)\,.$$

According to Theorem 1.3 [JDE 2021] in the case of source free equation, the solution to the Cauchy problem with $m \in \mathbb{C}$ is given by (9).

$\boxed{\text{For } m = 0}$ we obtain $K_1\,(r,t;0;\varepsilon) = \phi(\varepsilon)^{-1}$ and, consequently,

$$\begin{aligned} & \mathcal{K}_1(x,t,D_x;0;\varepsilon)[\varphi](x,t) \\ := \quad & -i\varepsilon^{\frac{\ell}{2}}(1-\ell)^{-1} \int_0^{\phi(t)-\phi(\varepsilon)} \int_{\mathbb{R}^3} \mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr. \end{aligned}$$

Hence the solution

$$\mathcal{K}_1(x,t,D_x;0;\varepsilon)[\varphi](x,t) = -i\varepsilon^{\frac{\ell}{2}}(1-\ell)^{-1} \int_{\mathbb{R}^3} \mathscr{V}^w(x-y,\phi(t)-\phi(\varepsilon))\varphi(y)\,dy,$$

obeys the Huygens' principle.

Proof of Sufficiency $m = \pm i(1-\ell)$

For $m = \pm i(1-\ell)$

$$K_1(r,t;i(1-\ell);\varepsilon) = r^2\left(\frac{1}{2}\ell^3\varepsilon^{3\ell-3} - \frac{3}{2}\ell^2\varepsilon^{3\ell-3} - \frac{1}{2}\varepsilon^{3\ell-3} + \frac{3}{2}\ell\varepsilon^{3\ell-3}\right) + \left(\frac{1}{2}\varepsilon^{3\ell-3}t^{2-2\ell} - \frac{1}{2}\ell\varepsilon^{3\ell-3}t^{2-2\ell} - \frac{1}{2}\ell\varepsilon^{\ell-1} + \frac{\varepsilon^{\ell-1}}{2}\right),$$

$$K_1(r,t;-i(1-\ell);\varepsilon) = -(\ell-1)t^{\ell-1}.$$

In fact, the family of more general $K_1(r,t;\pm m;\varepsilon)$ obeys the Huygens' principle.

Theorem

If the kernels $K_1(r,t;\pm m;\varepsilon)$ can be represented in the form

$$K_1(r,t;\pm m;\varepsilon) = r^2 a_\pm(\pm m;\varepsilon) + b_\pm(t;\pm m;\varepsilon),$$

then the Dirac equation (8) obeys the Huygens' principle.

According to (7) we have

$$
\begin{aligned}
&\mathcal{K}_1(x,t,D_x;m;\varepsilon)[\varphi](x,t) \qquad (10)\\
=\ & -i\varepsilon^{1+\frac{\ell}{2}-im}(1-\ell)^{-1}a_+(m;\varepsilon)\int_0^{\phi(t)-\phi(\varepsilon)} r^2\int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr\\
& -i\varepsilon^{1+\frac{\ell}{2}-im}(1-\ell)^{-1}b_+(t;m;\varepsilon)\int_0^{\phi(t)-\phi(\varepsilon)} \int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr.
\end{aligned}
$$

Then

$$
\int_0^{\phi(t)-\phi(\varepsilon)} \int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr = \int_{\mathbb{R}^3}\mathscr{V}^w(x-y,\phi(t)-\phi(\varepsilon))\varphi(y)\,dy.
$$

This term obeys the Huygens' principle since $\mathscr{V}^w$ does it. Next :

$$
\begin{aligned}
& \int_0^{\phi(t)-\phi(\varepsilon)} r^2\int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr\\
=\ & \left(r^2\int_{\mathbb{R}^3}\mathscr{V}^w(x-y,r)\varphi(y)\,dy\right)_{r=\phi(t)-\phi(\varepsilon)}\\
& -2\int_0^{\phi(t)-\phi(\varepsilon)} r\left(\int_{\mathbb{R}^3}\mathscr{V}^w(x-y,r)\varphi(y)\,dy\right)dr\,.
\end{aligned}
$$

Proof of Sufficiency $m = \pm i(1-\ell)$

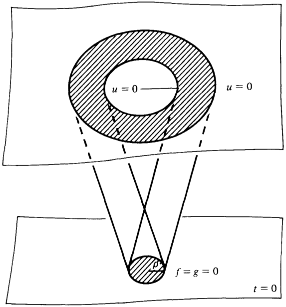
By the Kirchhoff's formula (up to unimportant factor):

$$
\begin{aligned}
& \frac{\partial}{\partial x_3} \int_0^{\phi(t)-\phi(\varepsilon)} r \left(\int_{\mathbb{R}^3} \mathscr{V}^w(x-y,r)\varphi(y)\,dy \right) dr \\
= & \int\!\!\int\!\!\int_{|y|\leqslant \phi(t)-\phi(\varepsilon)} \frac{\partial}{\partial y_3}\varphi(x+y)dy_1\, dy_2\, dy_3 \\
= & \int_{y_1^2+y_2^2\leqslant(\phi(t)-\phi(\varepsilon))^2} \left\{ \varphi\left(x_1+y_1, x_2+y_2, x_3+\sqrt{(\phi(t)-\phi(\varepsilon))^2-y_1^2-y_1^2} \right) \right. \\
& \left. -\varphi\left(x_1+y_1, x_2+y_2, x_3-\sqrt{(\phi(t)-\phi(\varepsilon))^2-y_1^2-y_1^2} \right) \right\} dy_1\, dy_2 \,.
\end{aligned}
$$

For every $t>0$ the points $\left(x_1+y_1, x_2+y_2, x_3 \pm \sqrt{(\phi(t)-\phi(\varepsilon))^2-y_1^2-y_1^2}\right) \in \mathbb{R}^3$, where $y_1^2+y_2^2 \leqslant (\phi(t)-\phi(\varepsilon))^2$, belong to the sphere of the radius $|\phi(t)-\phi(\varepsilon)|$. The domain of integration does not intersect the interior of the domain of dependence. Thus, the tail is empty. The sufficiency is proved.

Proof of Necessity $m = 0, \pm i(1-\ell)$

Step 1. The choice of the special initial spinor that is a radial spinor with a support in small neighborhood of the origin.
Step 2. Establishing asymptotic behavior of the solution for large time at the spatial origin. If the value of the solution at the spatial origin differs from zero for the large time, then the Huygens' principle is violated.



Proof of Necessity $m = 0, \pm i(1-\ell)$

Consider the solution of the Cauchy problem with the radial initial data $\Phi(x) = (\Phi_0(x), \Phi_1(x), \Phi_2(x), \Phi_3(x))^T$ will be chosen having small support.

According to Theorem 1.3 [JDE 2021], the solution to the Cauchy problem with $m \in \mathbb{C}$, is given by

$$\Psi(x,t) \quad = \quad \mathscr{D}^{co}(t, \partial_t, \partial_x)\gamma^0 \begin{pmatrix} \mathcal{K}_1(x,t,D_x; m; \varepsilon)[\Phi_0(x)] \\ \mathcal{K}_1(x,t,D_x; m; \varepsilon)[\Phi_1(x)] \\ \mathcal{K}_1(x,t,D_x; -m; \varepsilon)[\Phi_2(x)] \\ \mathcal{K}_1(x,t,D_x; -m; \varepsilon)[\Phi_3(x)] \end{pmatrix} (x,t), \quad t > \varepsilon > 0\,.$$

Proof of Necessity $m = 0, \pm i(1-\ell)$

Let $\Phi(x) = \Phi(r)$ be a spinor-valued function,
$\operatorname{supp}\Phi \subset \{x \in \mathbb{R}^n; |x| \le \min\{1/3, \widetilde{\varepsilon}/|\ell - 1|\}\}$, $\widetilde{\varepsilon} \in (0,1)$.

If we choose the initial data

$$\Phi(x) = (\Phi_0(x), 0, 0, 0)^T,$$

then the first component of $\Psi(x,t)$ is

$$\begin{aligned}
& \Psi_0(x,t) \\
= \; & it^{-\frac{\ell}{2}+im} \frac{\partial}{\partial t} \left(\mathcal{K}_1(x,t,D_x;m;\varepsilon)[\Phi_0(x)]\right)(x,t) \\
= \; & c_{\varepsilon,\ell,m} t^{-\frac{\ell}{2}+im} \frac{\partial}{\partial t} \left(\int_0^{\phi(t)-\phi(\varepsilon)} K_1(r,t;m;\varepsilon) \int_{\mathbb{R}^3} \mathcal{E}^w(x-y,r)\Phi_0(y)\, dy\, dr \right),
\end{aligned}$$

$t > \varepsilon > 0$.

Proof of Necessity $m = 0, \pm i(1-\ell)$

Denote $\varphi(y) := \Phi_0(y)$. In

$$\frac{\partial}{\partial t}\left(\int_0^{\phi(t)-\phi(\varepsilon)} K_1(r,t;m;\varepsilon)\int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\Phi_0(y)\,dy\,dr\right),$$

we need to find a large time asymptotics only for

$$\int_0^{\phi(t)-\phi(\varepsilon)}\left(\frac{\partial}{\partial t}K_1(r,t;\pm m;\varepsilon)\right)\int_{\mathbb{R}^3}\mathcal{E}^w(x-y,r)\varphi(y)\,dy\,dr,$$

since the term obtained by differentiation of the upper limit

$$t^{-\ell}K_1(\phi(t)-\phi(\varepsilon),t;\pm m;\varepsilon)\int_{\mathbb{R}^3}\mathcal{E}^w(x-y,\phi(t)-\phi(\varepsilon))\varphi(y)\,dy,$$

obeys the Huygens' principle. Hence, we consider

$$\int_0^{\phi(t)-\phi(\varepsilon)}\left(\frac{\partial}{\partial t}K_1(r,t;\pm m;\varepsilon)\right)\int_{\mathbb{R}^3}\partial_r\mathscr{V}^w(x-y,r)\varphi(y)\,dy\,dr.$$

Proof of Necessity $m = 0,\ \pm i(1-\ell)$

Denote

$$\mathscr{V}_\varphi(x,r) := \int_{\mathbb{R}^3} \mathscr{V}^w(x-y,r)\varphi(y)\,dy$$

and

$$\widetilde{\Psi}(x,t) := \int_0^{\phi(t)-\phi(\varepsilon)} \left(\frac{\partial}{\partial t}K_1\left(r,t;\pm m;\varepsilon\right)\right)\partial_r \mathscr{V}_\varphi(x,r)\,dr\,,$$

Then

$$\begin{aligned}\widetilde{\Psi}(x,t) &= \left(\frac{\partial}{\partial t}K_1\left(\phi(t)-\phi(\varepsilon),t;\pm m;\varepsilon\right)\right)\mathscr{V}_\varphi(x,\phi(t)-\phi(\varepsilon)) \\ &\quad -\int_0^{\phi(t)-\phi(\varepsilon)}\left(\frac{\partial}{\partial r}\frac{\partial}{\partial t}K_1\left(r,t;\pm m;\varepsilon\right)\right)\mathscr{V}_\varphi(x,r)\,dr\,.\end{aligned}$$

Proof of Necessity $m = 0, \pm i(1-\ell)$

By the Kirchhoff's formula we have $\mathscr{V}_\varphi(0,r) = r\varphi(r)$ and

$$\mathscr{V}_\varphi(0, \phi(t)-\phi(\varepsilon)) = (\phi(t)-\phi(\varepsilon))\varphi_0(\phi(t)-\phi(\varepsilon)) = 0$$

for sufficiently large t, that is, if $\phi(t)-\phi(\varepsilon) > \widetilde{\varepsilon}$. Consequently, for large t we have

$$\begin{aligned}
\widetilde{\Psi}(0,t) &= -\int_0^{\phi(t)-\phi(\varepsilon)} \left(\frac{\partial}{\partial r}\frac{\partial}{\partial t} K_1(r,t;\pm m;\varepsilon) \right) r\varphi(r)\, dr \\
&= \int_0^{\widetilde{\varepsilon}} \left(\frac{\partial}{\partial t} K_1(r,t;\pm m;\varepsilon) \right) \frac{\partial}{\partial r}(r\varphi(r))\, dr\,. \qquad (11)
\end{aligned}$$

Proof of Necessity $m = 0, \pm i(1-\ell)$

The outline of the remaining part of the proof is:

- We study the asymptotcs of $\frac{\partial}{\partial t} K_1(r,t;\pm m;\varepsilon)$ as $t \to \infty$. The principal term of this asymptotics for the cases of $\boxed{m \neq 0, \pm i(\ell-1)}$ is the function of r that allows us to find the initial function $\varphi(r)$ such that $\widetilde{\Psi}(0,t) \neq 0$ for sufficiently large time, and Huygens' principle is violated for such m.
- It suffices to study the asymptotic behavior of

$$\frac{\partial}{\partial \tau}\left(\left((\tau+1)^2 - A^2\right)^{-M} F\left(M, M; 1; \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2}\right)\right)$$

  for $M \in \mathbb{C}$, $M \neq 0, \pm 1$, as $\tau \to 0$ if $\ell > 1$ or as $\tau \to \infty$ if $\ell < 1$.
- Here $\tau = t^{1-\ell}$, while $A^2 := (\ell-1)^2 r^2$ is sufficiently small, and

$$0 < \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} < 1 \quad \text{when} \quad \tau \in (0, 1/2) \cup (2, \infty) \quad \text{and} \quad A^2 \in [0, 1/9],$$

$$\lim_{\tau \to 0} \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} = \lim_{\tau \to \infty} \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} = 1 \quad \text{uniformly on} \quad A^2 \in [0, 1/9]\,.$$

We use the finite sum of the asymptotic (as $t \to \infty$) series

$$\sum_{k=-\infty}^{\infty} \tau^{a_k} \left(h_k(A,M) + \widetilde{h}_k(A,M)(\ln(\tau))^{b_k} \right) ,$$

where $a_k \in \mathbb{C}$, $k = 0, \pm 1, \pm 2, \dots$, and

$$\dots < \Re a_{k-1} < \Re a_k < \Re a_{k+1} < \dots , \quad \lim_{k\to-\infty} \Re a_k = -\infty , \quad \lim_{k\to\infty} \Re a_k = \infty .$$

If $h_k(A,M)$ or $\widetilde{h}_k(A,M)$ is independent of A, then

$$\int_0^{\widetilde{\varepsilon}} h_k(A,M) \frac{\partial}{\partial r}(r\varphi(r))\, dr = 0 \quad \text{or} \quad \int_0^{\widetilde{\varepsilon}} \widetilde{h}_k(A,M) \frac{\partial}{\partial r}(r\varphi(r))\, dr = 0$$

for every $\varphi \in C_0^\infty(0,\widetilde{\varepsilon})$.
We show that all $h_k(A,M)$, $\widetilde{h}_k(A,M)$ are polynomials in $\sqrt{1-A^2}$ or $\ln(1-A^2)$ (if $\ell < 1$) or rational in $1-A^2$ (if $\ell > 1$) functions.
It is enough to prove the existence of the depending on A coefficient $h_k(A,M)$ or $\widetilde{h}_k(A,M)$. Then the existence of a function $\varphi \in C_0^\infty(0,\widetilde{\varepsilon})$ such that

$$\widetilde{\Psi}(0,t) = \int_0^{\widetilde{\varepsilon}} \left(\frac{\partial}{\partial t} K_1(r,t;\pm m;\varepsilon) \right) \frac{\partial}{\partial r}(r\varphi(r))\, dr \neq 0 \quad \text{for sufficiently large} \quad t ,$$

is evident. This completes the proof of Theorem.

Huygens' Principle and incomplete Huygens' principle for Klein-Gordon Equation in FLRW

- (KY'13) $m = \sqrt{n^2 - 1}/2$ is the only value of the mass such that
$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = 0,$$
obeys the Huygens' principle, whenever the wave equation in the Minkowski spacetime does, that is $n \geq 3$ is an odd number.
- If $n = 3$, then $m^2 = 2$
- We say that the equation obeys the incomplete Huygens' principle (KY'13) with respect to the first initial datum, if it obeys the Huygens' principle provided that the second initial datum $\Phi_t(t = 0) \equiv 0$.
- Suppose that $m^2 \neq 2$. Then, equation obeys the incomplete Huygens' principle if and only if $m = 0$, and either $n = 1$ or $n = 3$.
- Assume that the equations $\Phi_{tt} - e^{-2t}\Delta\Phi + n\Phi_t + m_k^2\Phi = 0, \quad k = 1, 2,$ describe two fields with $m_1 \neq m_2$.

  Then they obey the incomplete Huygens' principle if and only if the dimension n of the spatial variable x is 3 and $m_1 = 0$, $m_2 = \sqrt{2}$.

Huygens' Principle in FLRW Klein-Gordon and wave

KY [JMP 2013] for equations in flat de Sitter universe $a(t) = e^{Ht}$:

- Huygens' principle for Klein-Gordon equation $m^2 = 2H^2$
- incomplete Huygens' principle for wave equation $m = 0$

Peter Hintz, YuQing Xie: PHYSICAL REVIEW D, 2021

- Quasinormal modes and dual resonant states on de Sitter space for $m = 0$ discovered phenomenon which is closely related to incomplete Huygens principle

Jose Natario, Flavio Rossetti [arXiv 2021] for Wave Equation

- Radiation-filled flat universe: wave equation satisfies the strong Huygens principle
- Radiation-filled hyperbolic universe: strong Huygens principle
- Anti-de Sitter universe: incomplete Huygens principle
- Hyperbolic de Sitter universe: incomplete Huygens principle
- Radiation-filled spherical universe: strong Huygens principle
- Spherical de Sitter universe: incomplete Huygens principle

Thank you for your time!