Berezin-Toeplitz quantization on symplectic manifolds of bounded geometry

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Quantization problem

Classical mechanical system

- $(X, \mathbf{B})$ a symplectic manifold, $\dim X = 2n$.
- $C^\infty(X)$ algebra of classical observables.
- $f, g \in C^\infty(X) \mapsto \{f, g\} \in C^\infty(X)$ the Poisson bracket.

A quantization of $(X, \mathbf{B})$ is a construction of the corresponding quantum system:

A family of linear maps $f \in C^\infty(X) \mapsto T^\hbar(f) \in \mathcal{L}(\mathcal{H}^\hbar), \hbar > 0$
a Hilbert space $\mathcal{H}^\hbar$ is the quantum space:

- $\|T^\hbar(f) : \mathcal{H}^\hbar \to \mathcal{H}^\hbar\| = \|f\| + \mathcal{O}(\hbar), \hbar \to 0$,
  where $\|f\| = \sup_{x\in X} |f(x)|$ the norm of f in $C(X)$.
- $[T^\hbar(f), T^\hbar(g)] = i\hbar T^\hbar(\{f, g\}) + \mathcal{O}(\hbar^2), \hbar \to 0$,

$$\left\| [T^\hbar(f), T^\hbar(g)] - i\hbar T^\hbar(\{f, g\}) : \mathcal{H}^\hbar \to \mathcal{H}^\hbar \right\| \leqslant C\hbar^2.$$

Berezin-Toeplitz quantization

Berezin-Toeplitz quantization is a version of geometric quantization.

- Suggested by:
  F.A. Berezin,
  Quantization. Math. USSR-Izv. 8 (1974), 1109–1165.
  L. Boutet de Monvel and V. Guillemin,
  The spectral theory of Toeplitz operators, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ, 1981.
- For compact Kähler manifolds:
  Bordemann, M., Meinrenken, E., and Schlichenmaier, M.,
  Toeplitz quantization of Kähler manifolds and $gl(n)$; $n \to \infty$ limits. Commun. Math. Phys. 165 (1994), 281–296.

Geometric quantization (Kostant, Souriau (1970))

Assume a compact symplectic manifold $(X, \mathbf{B})$ is quantizable:

$$\frac{1}{2\pi\hbar}[\mathbf{B}] \in H^2(X, \mathbb{Z}).$$

Then there exists a Hermitian line bundle $(L_\hbar, h^{L_\hbar})$ on X with a Hermitian connection $\nabla^{L_\hbar}$ (a prequantum line bundle) such that

$$\frac{1}{\hbar}\mathbf{B} = iR^{L_\hbar},$$

$R^{L_\hbar} = (\nabla^{L_\hbar})^2$ is the curvature of the connection (a closed two-form).

Geometric quantization (Kostant, Souriau (1970))

Choose $L := L_1$ to satisfy the quantization condition for $\hbar = 1$.
Then its tensor power

$$L^p := L^{\otimes p} = L_\hbar, \quad p \in \mathbb{N}$$

satisfies the quantization condition for $\hbar = \frac{1}{p}, p \in \mathbb{N}$.
$\hbar = \frac{1}{p}$ takes a discrete set of values.
The semiclassical limit

$$\hbar \to 0 \Leftrightarrow p \to +\infty.$$

The quantum space $\mathcal{H}^\hbar = \mathcal{H}_p$ is a closed subspace of $L^2(X, L^p)$.

Kähler quantization

- $(X, \mathbf{B})$ a quantizable compact Kähler manifold,
  It means that there exists a complex structure and a Riemannian metric g on X such that $H = g + i\mathbf{B}$ is a Hermitian metric.
- The quantum line bundle L is a holomorphic line bundle with holomorphic connection (Chern connection).
- The quantum space $\mathcal{H}_p = Ker\,\bar\partial^{L^p}$ is the space of holomorphic sections of L^p.
- $P_{\mathcal{H}_p}$ is the Bergman projection in $L^2(X, L^p)$ onto $\mathcal{H}_p$.
- $\dim \mathcal{H}_p < \infty$ given by the Riemann-Roch-Grothendick theorem. In particular,

$$\dim \mathcal{H}_p \sim \frac{1}{(2\pi)^n} p^n \int_X \frac{\mathbf{B}^n}{n!}, \quad p \to \infty.$$

Kähler quantization

For $f \in C^\infty(X)$, the associated Toeplitz operator $T_{f,p} : \mathcal{H}_p \to \mathcal{H}_p$:

$$T_{f,p}u = P_{\mathcal{H}_p}(fu), \quad u \in \mathcal{H}_p,$$

or

$$T_{f,p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p} : L^2(X, L^p) \to L^2(X, L^p).$$

Theorem (M. Bordemann, E. Meinrenken, M. Schlichenmaier (1994))

For any $f, g \in C^\infty(X)$,

- $\|T_{f,p}\| = \|f\| + \mathcal{O}(p^{-1})$, $p \to \infty$.
- $[T_{f,p}, T_{g,p}] = ip^{-1} T_{\{f,g\},p} + \mathcal{O}(p^{-2})$, $p \to \infty$.

The proof is largely based on the theory of generalized Toeplitz structures developed in the mid-seventies by Boutet de Monvel, Guillemin, and Sjostrand in the framework of microlocal analysis.

Bochner Laplacian quantization

The initial idea of Bochner Laplacian quantization is due to Guillemin-Uribe, 1988.
Constructed for compact almost-Kähler manifolds by Borthwick-Uribe, 1996.

- $(X, \mathbf{B})$ a quantizable compact symplectic manifold.
- (L, h^L, ∇^L) a quantum line bundle.
- g a Riemannian metric on X.
- $\nabla^{L^p} : C^\infty(X, L^p) \to C^\infty(X, T^*X \otimes L^p)$ the induced connection.
- $(\nabla^{L^p})^* : C^\infty(X, T^*X \otimes L^p) \to C^\infty(X, L^p)$ its formal adjoint w. r. to the L^2-inner products on $C^\infty(X, L^p)$ and $C^\infty(X, T^*X \otimes L^p)$.
- The Bochner Laplacian is the second order differential operator

$$\Delta^{L^p} = (\nabla^{L^p})^* \nabla^{L^p} : C^\infty(X, L^p) \to C^\infty(X, L^p).$$

Renormalized Bochner Laplacian

- Let $B_x : T_xX \to T_xX$ be a skew-adjoint operator such that

$$\mathbf{B}_x(u,v) = g(B_x u, v), \quad u, v \in T_xX.$$

$$\sigma(B_x) = \{\pm i a_j(x), j = 1, \ldots, n\}, \quad a_j(x) > 0.$$

- $\tau(x) := \mathrm{Tr}^+(B_x) = \sum_{j=1}^n a_j(x)$.
- The renormalized Bochner Laplacian Δ_p acts on $C^\infty(X, L^p)$:

$$\Delta_p = \Delta^{L^p} - p\tau.$$

Kähler case

If $(X, \mathbf{B})$ a compact Kähler manifold, L a holomorphic line bundle, then

- $\Delta_p = 2\square^{L^p}$, where $\square^{L^p} = (\bar\partial^{L^p})^* \bar\partial^{L^p}$ is the Kodaira Laplacian on L^p.
- $\sigma(\Delta_p) \subset \{0\} \cup [p\mu - C, +\infty)$.
- The 0-eigenspace is the space of holomorphic sections of L^p.

Generalized Bergman projection

Theorem (Guillemin-Uribe, 1988; Ma-Marinescu, 2002)

There exist $C > 0$ and $\mu > 0$ such that for any p

$$\sigma(\Delta_p) \subset [-C, C] \cup [p\mu - C, +\infty).$$

- $\mathcal{H}_p$ the linear subspace of $L^2(X, L^p)$ spanned by the eigensections of Δ_p corresponding to eigenvalues in $[-C, C]$.
- $\dim \mathcal{H}_p \sim \frac{1}{(2\pi)^n} p^n \int_X \frac{\mathbf{B}^n}{n!}, \quad p \to \infty$.
- $P_{\mathcal{H}_p}$ the orthogonal projection in $L^2(X, L^p)$ onto $\mathcal{H}_p$ (generalized Bergman projection).
- For $f \in C^\infty(X)$, the Toeplitz operator is a sequence of operators:

$$T_{f,p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p} : L^2(X, L^p) \to L^2(X, L^p).$$

Algebra of Toeplitz operators

Theorem (YK 2017, Ioos-Lu-Ma-Marinescu 2017)

For any $f, g \in C^\infty(X)$, $T_{f,p}T_{g,p}$ admits the asymptotic expansion

$$T_{f,p}T_{g,p} \sim \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p},$$

with some $C_r(f,g) \in C^\infty(X)$, where C_r are bidifferential operators:

$$C_0(f,g) = fg, \quad C_1(f,g) - C_1(g,f) = i\{f,g\}.$$

Here for any natural k there exists $C_k > 0$ such that

$$\left\| T_{f,p}T_{g,p} - \sum_{r=0}^{k} p^{-r} T_{C_r(f,g),p} : L^2(X, L^p) \to L^2(X, L^p) \right\| \leqslant C_k p^{-k-1}.$$

Algebra of Toeplitz operators

- A formal star product on $C^\infty(X)$:

$$f *_p g = \sum_{r=0}^{\infty} p^{-r} C_r(f,g), \quad f, g \in C^\infty(X),$$

which is a main object in deformation quantization.

- The proof is based on asymptotic expansions of Bergman kernel outside the diagonal (Dai-Liu-Ma (2006), Ma-Marinescu (2008) for Spin^c Dirac quantization) using methods of local index theory (Bismut-Lebeau localization technique Bismut-Lebeau (1991)) combined with weighted estimates technique (Meladze-Shubin1983, YK1991).

Non-compact manifolds: assumptions

$(X, \mathbf{B})$ a symplectic manifold of dimension $2n$, satisfying the following assumptions:

X is endowed with a Riemannian metric g such that (X, g) is a Riemannian manifold of bounded geometry.

The curvature R^{TX} of the Levi-Civita connection ∇^{TX} of (X, g) and its derivatives of any order are uniformly bounded on X in the norm induced by g, and the injectivity radius r_X of (X, g) is positive.

In particular, the Riemannian manifold (X, g) is complete.

Examples

- Lie groups and homogeneous spaces.
- Regular coverings of compact manifolds.
- Leaves of compact foliated manifolds.

Non-compact manifolds: assumptions

$\mathbf{B}$ is uniformly C^∞-bounded on (X, g).

$\mathbf{B}$ and its covariant derivatives of any order are uniformly bounded on X in the norm induced by g.

$\mathbf{B}$ is uniformly non-degenerate on X.

Recall that $\sigma(B_x) = \{\pm i a_j(x), j = 1, \ldots, n\}, a_j(x) > 0$. Then

$$a_j(x) \geqslant \mu_0 > 0, \quad x \in X, \quad j = 1, \ldots, n.$$

$(X, \mathbf{B})$ is quantizable.

There exists a Hermitian line bundle (L, h^L) on X with Hermitian connection ∇^L such that that its curvature R^L satisfies the prequantization condition:

$$iR^L = \mathbf{B}.$$

Generalized Bergman projection

- Since (X,g) is complete, the renormalized Bochner Laplacian $\Delta_p = \Delta^{L^p} - p\tau$ acting on the space $C_c^\infty(X,L^p)$ is essentially self-adjoint in the Hilbert space $L^2(X,L^p)$.
- Δ_p the unique self-adjoint extension of Δ_p in $L^2(X,L^p)$.
- Spectral gap property: There exists $C_L>0$ such that for any $p\in\mathbb{N}$ the spectrum of Δ_p in $L^2(X,L^p)$ satisfies

$$\sigma(\Delta_p)\subset\big[-C_L,C_L\big]\cup\big[2p\mu_0-C_L,+\infty\big).$$

- $\mathcal{H}_p\subset L^2(X,L^p)$ the spectral subspace of Δ_p corresponding to $[-C_L,C_L]$ (given by the spectral theorem).
- $P_{\mathcal{H}_p}: L^2(X,L^p)\to\mathcal{H}_p$ the corresponding spectral projection of Δ_p (the generalized Bergman projection).
- Asymptotic expansions for the generalized Bergman projection (YK-Ma-Marinescu 2019)

Algebra of observables

$f \in C_b^\infty(X)$, if $f \in C^k(X)$ and

$$\|f\|_{C_b^k} = \sup_{x\in X, \ell\leqslant k} \left|\left(\nabla^{TX}\right)^\ell f(x)\right|_x < \infty.$$

where $|\cdot|_x$ is the norm in $(T_x^*X)^{\otimes\ell}$ defined by g

The Levi-Civita connection $\nabla^{TX} : C^\infty(X) \to C^\infty(X, T^*X)$ defines a Hermitian connection on each $(T^*X)^{\otimes j}$ for $j \in \mathbb{N}$:

$$\nabla^{TX} : C^\infty(X, (T^*X)^{\otimes j}) \to C^\infty(X, (T^*X)^{\otimes(j+1)}).$$

For every $\ell \in \mathbb{N}$, we have the operator

$$\left(\nabla^{TX}\right)^\ell : C^\infty(X) \to C^\infty(X, (T^*X)^{\otimes\ell}).$$

Berezin-Toeplitz quantization

The Berezin-Toeplitz quantization of $f \in C_b^\infty(X)$:

$$T_{f,p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p} : L^2(X, L^p) \to L^2(X, L^p), \quad p \in \mathbb{N}.$$

Theorem (Y.K., 2021)

If $f, g \in C_b^\infty(X)$, then, for the product of $\{T_{f,p}\}$ and $\{T_{g,p}\}$, we have

$$T_{f,p} T_{g,p} = T_{fg,p} + \mathcal{O}(p^{-1}), \quad p \to +\infty.$$

Moreover, for the commutator of $\{T_{f,p}\}$ and $\{T_{g,p}\}$, we have

$$[T_{f,p}, T_{g,p}] = ip^{-1} T_{\{f,g\},p} + \mathcal{O}(p^{-2}), \quad p \to +\infty.$$

Some history

For the complex n-dimensional vector space $\mathbb{C}^n$ with the standard symplectic form and Euclidean metric, the Berezin-Toeplitz quantization is given by the Toeplitz operators on the Fock space.

- It was discussed already by Berezin 1974, 1975.
- For functions from $C_b^{4n+6}(\mathbb{C}^n)$:
  Coburn, L.A.: Deformation estimates for Berezin-Toeplitz quantization. Commun. Math. Phys. 149, 415–424 (1992)
- Borthwick, D.: Microlocal techniques for semiclassical problems in geometric quantization. In: Perspectives on Quantization. Contemp. Mathematics, vol. 214, pp. 23–37. AMS (1998)
- For bounded, uniformly continuous functions on $\mathbb{C}^n$:
  Bauer, W., Coburn, L. A.: Uniformly continuous functions and quantization on the Fock space. Bol. Soc. Mat. Mex. (3) 22, 669–677 (2016)
  An example which indicates that the above theorem in general fails in the case of rapidly oscillating bounded functions.

Some history

The theory of Berezin-Toeplitz quantization was studied for various types of domains in $\mathbb{C}^n$

- Bauer, W., Coburn, L. A., Hagger, R.: Toeplitz quantization on Fock space. J. Funct. Anal. 274, 3531–3551 (2018)
- Bauer, W., Hagger, R., Vasilevski, N.: Uniform continuity and quantization on bounded symmetric domains. J. London Math. Soc., 96, 345–366 (2017)
- Borthwick, D., Lesniewski, A., Upmeier, H.: Non-perturbative deformation quantization of Cartan domains. J. Funct. Anal. 113, 153–176 (1993)
- Engliš, M.: Weighted Bergman kernels and quantization. Commun. Math. Phys. 227, 211–241 (2002)
- Klimek, S., Lesniewskii, A.: Quantum Riemann surfaces I: the unit disc. Commun. Math. Phys. 146, 103–122 (1992).

Non-compact manifolds

- Engliš, M., Upmeier, H.: Asymptotic expansions for Toeplitz operators on symmetric spaces of general type. Trans. Amer. Math. Soc. 367, 423–476 (2015)
  Berezin-Toeplitz quantization was constructed in the general setting of symmetric spaces of compact or noncompact type, both in the real and the complex hermitian case.
- Ma, X., Marinescu, G.: Toeplitz operators on symplectic manifolds. J. Geom. Anal. 18, 565–611 (2008)
  Berezin-Toeplitz quantization of the algebra $C^\infty_{const}(X)$ of smooth functions which are constant outside of a compact set on a class of complete Hermitian manifolds.

To our knowledge, there are no works on Berezin-Toeplitz quantization for a rather general class of noncompact symplectic manifolds.

Weighted L^2 spaces

- $L^2(X, L^p)$ the space of L^2-sections of L^p with the L^2-norm

$$\|u\|^2 = \int_X |u(x)|^2 dv_X(x), \quad u \in L^2(X, L^p),$$

where dv_X stands for the Riemannian volume form of (X, g).

- $\{d_y : y \in X\}$ the family of Lipschitz functions on X given by

$$d_y(x) = d(x, y), \quad x \in X,$$

$d(x, y)$ the geodesic distance on X.

- The family of weighted L^2-spaces

$$L^2_{\alpha,y}(X, L^p) = \{u \in C^{-\infty}(X, L^p) : e^{\alpha d_y} u \in L^2(X, L^p)\},$$
$$\|u\|_{p,\alpha,y} = \|e^{\alpha d_y} u\|, \quad u \in L^2_{\alpha,y}(X, L^p), \alpha \in \mathbb{R}, \quad y \in X,$$

As a topological vector space, $L^2_{\alpha,y}(X, L^p)$ is independent of y.

Toeplitz operators

A Toeplitz operator is a sequence of bounded linear operators

$$T_p : L^2(X, L^p) \to L^2(X, L^p), \quad p \in \mathbb{N},$$

satisfying the following conditions:
(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}.$$

(ii) There exists a sequence $g_l \in C_b^\infty(X,)$ such that

$$T_p \sim P_{\mathcal{H}_p} \left(\sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p}$$

in the following sense.

Toeplitz operators

(ii) There exists a sequence $g_l \in C_b^\infty(X)$ such that

$$T_p = P_{\mathcal{H}_p}\left(\sum_{l=0}^{\infty} p^{-l} g_l\right) P_{\mathcal{H}_p}$$

in the following sense: for any $K \in \mathbb{Z}_+$ denote the remainder

$$R_p^{(K)} = T_p - P_{\mathcal{H}_p}\left(\sum_{l=0}^{K} p^{-l} g_l\right) P_{\mathcal{H}_p}$$

There exist $\mu > 0$ and $C > 0$ such that for any $p \in \mathbb{N}$, $\alpha \in \mathbb{R}$ with $|\alpha| < \mu\sqrt{p}$ and $y \in X$,

$$\left\| R_p^{(K)} : L^2_{\alpha,y}(X, L^p) \to L^2_{\alpha,y}(X, L^p) \right\| \leqslant C p^{-K-1}.$$

Composition of Toeplitz operators

Theorem (Y.K., 2021)

Let $f, g \in C_b^\infty(X)$. Then the composition of the Toeplitz operators $T_{f,p}$ and $T_{g,p}$ is a Toeplitz operator in the above sense.
More precisely, it admits the asymptotic expansion

$$T_{f,p}T_{g,p} \sim \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p},$$

with some $C_r(f,g) \in C_b^\infty(X)$, where C_r are bidifferential operators. In particular, $C_0(f,g) = fg$ and, for $f, g \in C_b^\infty(X)$,

$$C_1(f,g) - C_1(g,f) = i\{f,g\}.$$

Characterization of Toeplitz operators

- The key ingredient in the proof is a characterization of Toeplitz operators in terms of asymptotic expansions of their Schwartz kernels.
- For compact manifolds, this type of characterization was introduced by Ma-Marinescu,2008 for Toeplitz operators associated with spin^c Dirac quantization and extended to Toeplitz operators associated with the renormalized Bochner Laplacian by YK, 2017 and Ioos-Lu-Ma-Marinescu, 2017.

Theorem

A sequence of bounded linear operators $T_p : L^2(X, L^p) \to L^2(X, L^p)$, is a Toeplitz operator in the above sense if and only if it satisfies the following three conditions:
(i) For any $p \in \mathbb{N}$,

$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}.$$

Characterization of Toeplitz operators

(ii) T_p has smooth Schwartz kernel $T_p(\cdot,\cdot) \in C^\infty(X \times X)$ with respect to the Riemannian volume form dv_X:

$$T_p s(x) = \int_X T_p(x,x')\, s(x')\, dv_X(x'),$$

and there exists $\mu > 0$ such that for any $\epsilon_0 > 0$, we have

$$\left| T_p(x,x') \right| \leqslant C e^{-\mu\sqrt{p}\, d(x,x')}, \quad p \in \mathbb{N}, \quad x, x' \in X, \quad d(x,x') > \epsilon_0,$$

where $C > 0$ depends only on ϵ_0.

Characterization of Toeplitz operators

- We will use the normal coordinates near an arbitrary point $x_0 \in X$.
- We denote by $B^X(x_0, r)$ and $B^{T_{x_0}X}(0, r)$ the open balls in X and $T_{x_0}X$ with center x_0 and radius r, respectively.
  $B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X)$
  via the exponential map $\exp^X_{x_0} : T_{x_0}X \to X$.
- Furthermore, we choose a trivialization of L over $B^X(x_0, r_X)$, identifying its fiber L_Z at $Z \in B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X)$ with the space L_{x_0} by parallel transport with respect to the connections ∇^L along the curve $\gamma_Z : [0, 1] \ni u \to \exp^X_{x_0}(uZ)$.
- The kernel $T_p(x, x')$ induces a family of smooth functions

$$T_{p,x_0}(Z, Z') = T_p(\exp^X_{x_0}(Z), \exp^X_{x_0}(Z')),$$
$$x_0 \in X, \quad Z, Z' \in B^{T_{x_0}X}(0, r_X).$$

Characterization of Toeplitz operators

(iii) There exists $\varepsilon' \in (0, r_X/4)$ such that, for any $k \in \mathbb{N}$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$,

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}),$$

where $\mathcal{Q}_{r,x_0} \in \mathbb{C}[Z, Z']$ is a family of polynomials, depending smoothly and C^∞-boundedly on x_0, of the same parity as $r \in \mathbb{Z}_+$ and $\mathcal{P}_{x_0} \in C^\infty(T_{x_0}X \times T_{x_0}X)$ is the Bergman kernel in $T_{x_0}X$.

Bergman kernel

If we choose an orthonormal base $\{e_j : j = 1, \dots, 2n\}$ in $T_{x_0}X$ such that

$$B_{x_0} e_{2k-1} = a_k e_{2k}, \quad B_{x_0} e_{2k} = -a_k e_{2k-1}, \quad k = 1, \dots, n,$$

then the Bergman kernel $\mathcal{P}_{x_0}$ is given by

$$\mathcal{P}_{x_0}(Z, Z') = \frac{1}{(2\pi)^n} \prod_{j=1}^{n} a_j \exp\left(-\frac{1}{4} \sum_{k=1}^{n} a_k (|z_k|^2 + |z'_k|^2 - 2 z_k \bar{z}'_k) \right),$$

$$Z, Z' \in \mathbb{R}^{2n} \cong T_{x_0}X,$$

where we use the complex coordinates $z_k = Z_{2k-1} + iZ_{2k}, k = 1, \dots, n$.

Characterization of Toeplitz operators

$$p^{-n}T_{p,x_0}(Z,Z') \cong \sum_{r=0}^{k}(Q_{r,x_0}\mathcal{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}})$$

means that there exist $\varepsilon' \in (0, r_X]$, $C_0 > 0$ and $c_0 > 0$ with the following property: for any $l \in \mathbb{N}$, there exist $C > 0$ and $M > 0$ such that for any $x_0 \in X$, $p \geqslant 1$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$, we have

$$\left|p^{-n}T_{p,x_0}(Z,Z')\kappa_{x_0}^{\frac{1}{2}}(Z)\kappa_{x_0}^{\frac{1}{2}}(Z') - \sum_{r=0}^{k}(Q_{r,x_0}\mathcal{P}_{x_0})(\sqrt{p}Z,\sqrt{p}Z')p^{-\frac{r}{2}}\right|_{C_b^l(X)} \leqslant Cp^{-\frac{k+1}{2}}(1+\sqrt{p}|Z|+\sqrt{p}|Z'|)^M e^{-C_0\sqrt{p}|Z-Z'|} + \mathcal{O}(e^{-c_0\sqrt{p}}).$$

Here $C_b^l(X)$ is the C_b^l-norm for the parameter $x_0 \in X$.

Characterization of Toeplitz operators

κ_{x_0} a smooth positive function on $B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X)$ defined by

$$dv_X(Z) = \kappa_{x_0}(Z) dv_{T_{x_0}X}(Z), \quad Z \in B^{T_{x_0}X}(0, r_X),$$

where
dv_X is the Riemannian volume form of the tangent space X equipped with the Riemannian metric g, and
$dv_{T_{x_0}X}$ is the Riemannian volume form of the tangent space $T_{x_0}X$ equipped with the Riemannian metric $g^{T_{x_0}X}$ defined by g.

Characterization of Toeplitz operators

Theorem

A family $\{T_p : L^2(X, L^p) \to L^2(X, L^p)\}$ *of bounded linear operators is a Toeplitz operator in the above sense if and only if it satisfies the following three conditions:*

(i) For any $p \in \mathbb{N}$,

$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}.$$

(ii) There exists $\mu > 0$ *such that for any* $\epsilon_0 > 0$, *we have*

$$\left|T_p(x, x')\right| \leqslant C e^{-\mu\sqrt{p}\, d(x,x')}, \quad p \in \mathbb{N}, \quad x, x' \in X, \quad d(x, x') > \epsilon_0.$$

(iii) There exists $\varepsilon' \in (0, r_X/4)$ *such that, for any* $k \in \mathbb{N}$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$,

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}),$$