Elliptic boundary problems for edge-degenerate pseudodifferential operators

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Let X be a $\mathscr{C}^\infty$ manifold with boundary, $\partial X = Y \neq \emptyset$, and $\dim X \geq 2$. Edge-degenerate ΨDOs on X are of the form

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where $t \geq 0$ is a boundary defining function, $y \in Y$, and m is the order of A.

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where $t \geq 0$ is a boundary defining function, $y \in Y$, and m is the order of A.

- The principal symbol of A is the pair $\left(\sigma_\psi^m(A), \sigma_\partial^m(A)\right)$, where

$$\sigma_\psi^m(A)(t, y, \tau, \eta) = a_{(m)}(t, y, \tau, \eta), \quad (t, y, \tau, \eta) \in T^*X \setminus 0,$$
$$\sigma_\partial^m(A)(y, \eta) = t^{-m} a(0, y, tD_t, t\eta), \quad (y, \eta) \in T^*Y \setminus 0.$$

- Note that $\sigma_\partial^m(A)(y, \eta)$ is a cone-degenerate ΨDOs on the half-line $[0, \infty)$.

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Example

ΨDOs on X with coefficients smooth up to ∂X are edge-degenerate.

Main objective

Suppose that A is elliptic in the sense that $\sigma_\psi^m(A)(t, y, \tau, \eta) \neq 0$ for all $(t, y, \tau, \eta) \in T^*X \setminus 0$. Our goal is to impose conditions of trace or potential type along Y so that the resulting boundary problem is Fredholm between suitable function spaces.

Main objective

Suppose that A is elliptic in the sense that $\sigma_\psi^m(A)(t, y, \tau, \eta) \neq 0$ for all $(t, y, \tau, \eta) \in T^*X \setminus 0$. Our goal is to impose conditions of trace or potential type along Y so that the resulting boundary problem is Fredholm between suitable function spaces.

- Notice that the ellipticity of A implies that $\sigma_\partial^m(A)(y, \eta)$ is Fredholm for all $(y, \eta) \in T^*Y \setminus 0$.
- Thus, on a symbolic level, the task is to add conditions of trace or potential type in a way that the "enlarged" principal boundary symbol becomes invertible.

To understand, what kind of trace or potential conditions can be added we need to look into the asymptotics of solutions to elliptic problems.

These asymptotics are (formally) of the form

$$u(t,y) \sim \sum_{(p,k)} \frac{(-1)^k}{k!} t^{-p} \log^k t \, u_{pk}(y) \quad \text{as } t \to +0.$$

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Now one of the ideas is to fix an asymptotic type P, i.e., to limit the pairs $(p,k) \in \mathbb{C} \times \mathbb{N}_0$ admitted in the infinite series.

- If u has asymptotics of type P, then Au has asymptotics of type R, for some resulting asymptotic type $R = \mathscr{R}(P,A)$.
- Given P, Q, we shall identify a class of

edge-degenerate ΨDOs A with $\mathscr{R}(P,A) \preccurlyeq Q$.

- The coefficients $\gamma_{pk} u = u_{pk}$ in the asymptotic expansion above are uniquely determined. Thus they define operators γ_{pk} which serve as prototypical examples of trace operators.

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Consequently, a boundary problem for an edge-degenerate operator A could be

$$\boxed{\begin{aligned} Au &= f(x), \qquad x \in X \setminus Y, \\ \gamma_{pk} u &= g_{pk}(y), \quad y \in Y, \ \ (p,k) \in P_0, \end{aligned}}$$

where $P_0 \subseteq P$, $\# P_0 < \infty$, and f has asymptotics of type Q.

Basic observation (B.-W. Schulze)

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- Our calculus $\Psi^{m,d,\delta}_{P,Q}(X;E,F)$ – with $P,Q \in \underline{\mathrm{As}}^{\delta}$ given – is modeled after Boutet de Monvel's calculus.
- Boutet de Monvel's calculus is then (essentially) the case $P = Q = P_0$ (and $\delta = 0$), where the asymptotic type $P_0 = \{(-\ell, 0) \mid \ell \in \mathbb{N}_0\}$ results from a Taylor series expansion at $t = 0$.

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In the rest of the talk, we shall mainly indicate the differences with Schulze's general edge calculus (where the type $d = 0$).

Previous work

- Calculus for pseudodifferential boundary problems: L. Boutet de Monvel (1971). See also the monographs by S. Rempel and B.-W. Schulze (1985), G. Grubb (1996), B.-W. Schulze (1998).
- Edge-degenerate pseudodifferential operators: S. Rempel and B.-W. Schulze (1989), B.-W. Schulze (1991).
- Construction of the conormal symbols: I. Witt (2002, 2007).
- Corresponding cone calculus ($\dim X = 1$): X.-C. Liu (2000), X.-C. Liu and I. Witt (2004).
- Function spaces $H^{s,\delta}_{P,\theta}(X)$: Z.-P. Ruan and I. Witt (preprint 2021).

Asymptotic types (indeed, constant discrete asymptotic types in Schulze's classification scheme) are non-negative divisors $p \mapsto m_p$ (in the sense of complex analysis) possessing additional properties.

It is often convenient to identify such a divisor with the set $P = \{(p, k) \in \mathbb{C} \times \mathbb{N}_0 \mid k < m_p\}$.

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Definition

Let $\delta \in \mathbb{R}$ (a reference conormal order). Then P as above belongs to $\underline{\mathrm{As}}^{\delta}$ if

1. the set $\pi_{\mathbb{C}} P = \{p \in \mathbb{C} \mid m_p > 0\}$ is contained in the half-space $\{z \in \mathbb{C} \mid \Re z < 1/2 - \delta\}$,
2. $\Re p \to -\infty$ as $|p| \to \infty$, $p \in \pi_{\mathbb{C}} P$,
3. $m_{p-1} \geq m_p$ for all p (for coordinate invariance).

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Example

$P_0 = \{(-\ell, 0) \mid \ell \in \mathbb{N}_0\} \in \underline{\mathrm{As}}^0$.

Recall the following facts:

1. $A = t^{-m} a(t, y, tD_t, tD_y)$ possesses the full sequence of conormal symbols $\{\sigma_{\mathrm{c}}^{m-j}(A)\}_{j \geq 0}$, where

$$\sigma_{\mathrm{c}}^{m-j}(A)(y, z, \eta) = \frac{1}{j!} \partial_t^j \left[a(t, y, \mathrm{i}z, t\eta)\right]\Big|_{t=0}$$

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   $\sigma_c^{m-j}(A)$ is a meromorphic function of $z \in \mathbb{C}$ that depends smoothly on y and is polynomial in η of degree j.

2. $\{\sigma_c^{m-j}(A)\}_{j\geq 0}$ controls the way in which asymptotics are mapped, i.e., it yields the aforementioned map $P \mapsto \mathscr{R}(P, A)$.
3. Composition is regulated by the Leibniz-Mellin translation product,

$$\sigma_c^{m+m'-l}(AB)(y, z, \eta) = \sum_{\substack{j+k=l,\\ |\alpha|\leq j}} \frac{1}{\alpha!}\big(\partial_\eta^\alpha \sigma_c^{m-j}(A)\big)(y, z+m'-k, \eta)\big(D_y^\alpha \sigma_c^{m'-k}(B)\big)(y, z, \eta).$$

Write $z = \beta + i\sigma$ and introduce the class $\mathcal{M}_P^\mu$ of all meromorphic functions $m(z)$ on $\mathbb{C}$ such that

1. $m \in \mathscr{C}^\infty(\mathbb{R}_\beta; S^\mu_{\mathrm{cl}}(\mathbb{R}_\sigma))$ “outside the poles of m”,
2. the principal divisor of $m(z)$ is bounded below by $-P$.

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Lemma

For $P \in \underline{\mathrm{As}}^\delta$, there are meromorphic functions $h_P \in \mathscr{C}^\infty(\mathbb{R}_\beta; \mathscr{S}(\mathbb{R}_\sigma))$ "outside the poles of h_P" such that

1. *the principal divisor of $h_P(z)$ equals $-P$,*
2. $\dfrac{h_Q(z+\mu)}{h_P(z)} \in \mathcal{M}_{\mathrm{as}}^\mu$ *for all $P, Q \in \underline{\mathrm{As}}^\delta$ and all $\mu \in \mathbb{R}$.*

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Example

$\Gamma(z)$ is an admissible choice for $h_{P_0}(z)$.

Theorem

Let A be edge-degenerate and of order m. Suppose that

$$\sigma_{\mathrm{c}}^{m-j}(A)(y,z,\eta) = \frac{h_Q(z+m-j)}{h_P(z)} \sum_{|\alpha|\le j} a_{j\alpha}(y,z)\eta^\alpha, \quad j \ge 0,$$

where $a_{j\alpha}$ is smooth in y with values in $\mathcal{M}^j_{\mathcal{O}}$ with $\mathcal{O}$ denoting the empty asymptotic type. Then:

1. $A\colon \mathscr{C}^\infty_P(X) \to \mathscr{C}^\infty_Q(X)$ *continuously,*
2. *If B is as above, but with asymptotic types Q, R, then the composition BA is as above, but with asymptotic types P, R.*

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A first answer

Let $\Psi_{P,Q}^{m,0,\delta}(X)$ consist of all edge-degenerate pseudodifferential operators with conormal symbols as given above. This choice is (a) independent of $h_P(z)$, $h_Q(z)$ and (b) coordinate-invariant.

From now on, we discuss the model case $X = \overline{\mathbb{R}}_{+}^{1+q}$. Note that our constructions are coordinate-invariant.

Definition (Weighted Sobolev spaces)

1. For $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_{+}^{1+q})$ consists of all u such that

$$t^{-\gamma}(tD_t)^j D_y^\alpha u \in L^2(\mathbb{R}_{+}^{1+q}), \quad j + |\alpha| \le s.$$

For general $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$, these spaces are then defined by interpolation and duality.

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   For general $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$, these spaces are then defined by interpolation and duality.

2. For $s \in \mathbb{R}$, $\gamma \in \mathbb{R}$, the space $\mathcal{K}^{s,\gamma}(\mathbb{R}_+^{1+q})$ consists of all u such that $\varphi(t)u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+^{1+q})$ and $(1-\varphi(t))\,u \in H^s(\mathbb{R}_+^{1+q})$. Here, φ is a cut-off function.

Denote by $H^{s,\langle k\rangle}(\mathbb{R}^q)$ for $s \in \mathbb{R}$, $k \in \mathbb{Z}$ the space of all $w = w(y)$ such that $\langle\eta\rangle^s \log^k\langle\eta\rangle \hat{w}(\eta) \in L^2(\mathbb{R}^q)$. Here, $\langle\eta\rangle = \left(2 + |\eta|^2\right)^{1/2}$.

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Definition (Potential operators, provisional)

For $(p, k) \in P$, define

$$(\Gamma_{pk} w)(t, y) = \frac{(-1)^k}{k!} \mathscr{F}^{-1}_{\eta\to y} \{\varphi(t\langle\eta\rangle)\hat{w}(\eta)\} \, t^{-p} \log^k t.$$

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Note that $\Gamma_{pk} w \in \mathscr{C}^\infty(\mathbb{R}^{1+q}_+)$ and $\operatorname{supp}(\Gamma_{pk} w) \subseteq \{t \le c\}$ for some $c > 0$.

Lemma

For $w \in H^{s,\langle k\rangle}(\mathbb{R}^q)$, $\Gamma_{pk} w \in \bigcap_{\epsilon>0} \mathcal{K}^{s+\epsilon,1/2-\Re p-\epsilon}(\mathbb{R}^{1+q}_+)$,

$$\Gamma_{pk} w - \varphi(t)t^{-p} \log^k t\, w(y) \in \bigcap_{\epsilon>0} \mathcal{K}^{s-\epsilon,1/2-\Re p+\epsilon}(\mathbb{R}^{1+q}_+).$$

Definition (Sobolev spaces with asymptotics, provisional)

Let $s \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^{\delta}$, $\theta \geq 0$. For $\pi_{\mathbb{C}} P \cap \{z \in \mathbb{C} \mid \Re z = 1/2 - \delta - \theta\} = \emptyset$, the space $H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+q}_{+})$ consists of all $u \in \mathcal{K}^{s,\delta}(\mathbb{R}^{1+q}_{+})$ for which there are functions $u_{pk} \in H^{s+\Re p+\delta-1/2,\langle k\rangle}(\mathbb{R}^q)$ such that

$$u(t,y) - \sum_{\substack{(p,k)\in P,\\ \Re p > 1/2-\delta-\theta}} (\Gamma_{pk} u_{pk})(t,y) \in \mathcal{K}^{s-\theta,\delta+\theta}(\mathbb{R}^{1+q}_{+}).$$

For general $\theta \geq 0$, $H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}^{1+q}_{+})$ is defined by interpolation.

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Remark

Later we shall enlarge the space of admissible traces u_{pk} slightly if $m_p > 1$ for some $p \in \pi_{\mathbb{C}} P$.

The basic case is $s \geq 0$ and $\theta = s$. In this case, we write

$$H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+q}) = H_{P,s}^{s,\delta}(\overline{\mathbb{R}}_+^{1+q}).$$

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Example

1. $H^s(\mathbb{R}_+^{1+q}) = H_{P_0}^{s,0}(\overline{\mathbb{R}}_+^{1+q})$.
2. $H_0^s(\overline{\mathbb{R}}_+^{1+q}) = H_{\mathcal{O}}^{s,0}(\overline{\mathbb{R}}_+^{1+q})$.

Definition

The class $\Psi^{m,d,\delta}_{P,Q}(\overline{\mathbb{R}}_+;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ consists of all operators

$$\mathfrak{A}=\begin{pmatrix} A+G & K \\ T & Q \end{pmatrix}:\begin{matrix} H^{s,\delta}_P(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_-}\end{matrix} \longrightarrow \begin{matrix} H^{s-m,\delta}_Q(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{N_+}\end{matrix},$$

for $s \geq m^+$ and $s > s_d$ when $d \geq 1$ such that $A = t^{-m}a(t,tD_t)$ is a cone-degenerate pseudodifferential operator of order m with conormal symbols

$$\sigma_c^{m-j}(A)(z)=\frac{h_Q(z+m-j)}{h_P(z)}\,a_j(z),$$

where $a_j \in \mathcal{M}^j_{\mathcal{O}}$ for $j \in \mathbb{N}_0$, and G, K, T, Q are accordingly.

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Main differences concerning function spaces and mapping properties:

- $u - \sum_{\Re p > 1/2-\delta-\theta} \frac{(-1)^k}{k!} \varphi(t) t^{-p} \log^k t \, u_{pk} \in \mathcal{K}^{s,\delta+\theta}(\mathbb{R}_+)$ (Schulze),
- $u - \ldots \in \mathcal{K}^{s-\theta,\delta+\theta}(\mathbb{R}_+)$ (Liu-R.-Witt),

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- $t^{-m} a(t, tD_t) \colon H^{s,\delta}_{P,\theta}(\overline{\mathbb{R}}_+) \to H^{s-m,\delta-m}_{Q,\theta}(\overline{\mathbb{R}}_+)$ (Schulze),
- $\ldots \colon H^{s,\delta}_{P,s}(\overline{\mathbb{R}}_+) \to H^{s-m,\delta}_{Q,s-m}(\overline{\mathbb{R}}_+)$ (Liu-R.-Witt).

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Let E be a Hilbert space and $\{\kappa_\lambda\}_{\lambda>0}$ be a strongly continuous representation of $\mathbb{R}_+$ on E. We shall write $\kappa(\eta)$ in place of $\kappa_{\langle\eta\rangle}$.

Definition (B.-W. Schulze)

For $s \in \mathbb{R}$, the abstract edge Sobolev space $\mathcal{W}^s(\mathbb{R}^q; E)$ consists of all $u \in \mathscr{S}'(\mathbb{R}^q; E)$ such that

$$\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q;E)} = \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \|\kappa(\eta)^{-1}\mathscr{F}u(\eta)\|^2_E \,\mathrm{d}\eta < \infty.$$

Here are two examples:

1. Change the definition of $H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+q})$ by demanding $(1-\varphi)\,u \in t^\delta H^s(\mathbb{R}_+^{1+d})$ (instead of $(1-\varphi)\,u \in H^s(\mathbb{R}_+^{1+d})$) and possibly enlarge the trace space according to the next item. Then

$$H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+q}) = \mathcal{W}^s(\mathbb{R}^q; H_P^{s,\delta}(\overline{\mathbb{R}}_+)),$$

where $(\kappa_\lambda v)(t) = \lambda^{1/2-\delta} v(\lambda t)$ for $\lambda > 0$.

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where $(\kappa_\lambda v)(t) = \lambda^{1/2-\delta} v(\lambda t)$ for $\lambda > 0$.

2. Let $m \in \mathbb{N}$ and introduce $\pi_\lambda \in \mathcal{L}(\mathbb{C}^m)$ for $\lambda > 0$ as the $m \times m$ upper triangular matrix with $(\pi_\lambda)_{jk} = \frac{(-1)^{k-j}}{(k-j)!} \log^{k-j}\lambda$ for $j \le k$. Then the trace space for $\gamma_p = (\gamma_{p0}, \gamma_{p1}, \dots, \gamma_{p,m_p-1})$ with $p \in \pi_{\mathbb{C}} P$ when acting on $\mathcal{W}^s(\mathbb{R}^q; H_P^{s,\delta}(\overline{\mathbb{R}}_+))$ is $H_\pi^{s'}(\mathbb{R}^q; \mathbb{C}^m) = \mathcal{W}^{s'}(\mathbb{R}^q; (\mathbb{C}^m, \{\pi_\lambda\}_{\lambda>0}))$, where $s' = s + \Re p + \delta - 1/2$.

Here are two examples:

1. Change the definition of $H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+q})$ by demanding $(1-\varphi)\,u \in t^\delta H^s(\mathbb{R}_+^{1+d})$ (instead of $(1-\varphi)\,u \in H^s(\mathbb{R}_+^{1+d})$) and possibly enlarge the trace space according to the next item. Then

$$H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+q}) = \mathcal{W}^s(\mathbb{R}^q; H_P^{s,\delta}(\overline{\mathbb{R}}_+)),$$

   where $(\kappa_\lambda v)(t) = \lambda^{1/2-\delta} v(\lambda t)$ for $\lambda > 0$.

2. Let $m \in \mathbb{N}$ and introduce $\pi_\lambda \in \mathcal{L}(\mathbb{C}^m)$ for $\lambda > 0$ as the $m \times m$ upper triangular matrix with $(\pi_\lambda)_{jk} = \frac{(-1)^{k-j}}{(k-j)!} \log^{k-j}\lambda$ for $j \le k$. Then the trace space for $\gamma_p = (\gamma_{p0}, \gamma_{p1}, \ldots, \gamma_{p,m_p-1})$ with $p \in \pi_{\mathbb{C}} P$ when acting on $\mathcal{W}^s(\mathbb{R}^q; H_P^{s,\delta}(\overline{\mathbb{R}}_+))$ is $H_\pi^{s'}(\mathbb{R}^q; \mathbb{C}^m) = \mathcal{W}^{s'}(\mathbb{R}^q; (\mathbb{C}^m, \{\pi_\lambda\}_{\lambda>0}))$, where $s' = s + \Re p + \delta - 1/2$.

   Note that

$$H^s(\mathbb{R}^q) \oplus H^{s,\langle 1\rangle}(\mathbb{R}^q) \oplus \ldots \oplus H^{s,\langle m-1\rangle}(\mathbb{R}^q) \subseteq H_\pi^s(\mathbb{R}^q; \mathbb{C}^m).$$

Not only function spaces are needed, but also operators.

Definition (B.-W. Schulze)

Let $E, \widetilde{E}$ be Hilbert spaces with strongly continuous group actions $\{\kappa_\lambda\}$ and $\{\widetilde{\kappa}_\lambda\}$. Further let $a\colon \mathbb{R}^q \times \mathbb{R}^q \to \mathcal{L}(E, \widetilde{E})$ be $\mathscr{C}^\infty$ for the strong operator topology. Then a is said to belong to $S^m(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$ if, for any α, β,

$$\left\|\widetilde{\kappa}(\eta)^{-1}(\partial_y^\alpha \partial_\eta^\beta a)(y, \eta)\kappa(\eta)\right\|_{E\to\widetilde{E}} \lesssim \langle\eta\rangle^{m-|\beta|}$$

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Analogously, we say that $a\colon \mathbb{R}^q \times (\mathbb{R}^q \setminus 0) \to \mathcal{L}(E, \widetilde{E})$ that is $\mathscr{C}^\infty$ for the strong operator topology belongs to $S^{(m)}(\mathbb{R}^q \times (\mathbb{R}^q \setminus 0); E, \widetilde{E})$ if

$$a(y, \lambda\eta) = \lambda^m \widetilde{\kappa}_\lambda a(y,\eta)\kappa_\lambda^{-1}, \quad \lambda > 0.$$

Then $\chi(\eta)a \in S^m(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$, where $\chi \in S^0(\mathbb{R}^q)$, $\chi(\eta) = 0$ for $|\eta| \lesssim 1$.

Lemma

For any $p \in \pi_{\mathbb{C}} P$,

$$\gamma_p = \lambda^{1/2-p-\delta} \pi_\lambda \gamma_p \kappa_\lambda^{-1}, \quad \lambda > 0.$$

In particular, $\gamma_p \in S^{1/2-\Re p-\delta}(\mathbb{R}^q \times \mathbb{R}^q; H_P^{s,\delta}(\overline{\mathbb{R}}_+), \mathbb{C}^m)$ provided that $s > 1/2 - \Re p - \delta$.

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Proposition (J. Seiler, 1999)

Let $a \in S^m(\mathbb{R}^q \times \mathbb{R}^q; E, \widetilde{E})$. Then

$$a(y, D_y) : \mathcal{W}^s(\mathbb{R}^q; E) \to \mathcal{W}^{s-m}(\mathbb{R}^q; \widetilde{E})$$

continuously for any $s \in \mathbb{R}$.

Symbol calculus means among other things to define a class

$$\mathfrak{a}(y,\eta) = \begin{pmatrix} a(y,\eta)+g(y,\eta) & k(y,\eta) \\ t(y,\eta) & q(t,\eta) \end{pmatrix}$$

of operator functions that take values in $\Psi^{m,d,\delta}_{P,Q}(\overline{\mathbb{R}}_+;\mathbb{C}^{N_-},\mathbb{C}^{N_+})$ and such that

$$\mathfrak{a} \in S^m(\mathbb{R}^q \times \mathbb{R}^q; H^{s,\delta}_P(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_-}, H^{s-m,\delta}_Q(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{N_+})$$

for $s \geq m^+$ and $s > s_d$ if $d \geq 1$.

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Let $f \in S^{m-\delta+1/2}(\mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}^{m_p}, \mathbb{C}) \hat{\otimes} \mathscr{S}_Q(\overline{\mathbb{R}}_+)$ and set

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$$k(y,\eta)c = f(y,\eta,t\langle\eta\rangle)c$$

for $c \in \mathbb{C}^{m_p}$. Then

$$\kappa(\eta)^{-1} f(y,\eta,t\langle\eta\rangle)\pi(\eta)c = \langle\eta\rangle^{\delta-1/2} f(y,\eta,t)\pi(\eta)c$$

so that

$$\|\kappa(\eta)^{-1} k(y,\eta)\pi(\eta)\|_{\mathbb{C}^{m_p} \to H_Q^{s-m,\delta}(\overline{\mathbb{R}}_+)} \lesssim \langle\eta\rangle^m,$$

similar for $\partial_y^\alpha \partial_\eta^\beta k(y,\eta)$.

Lemma

Let $p \in \pi_{\mathbb{C}} P$ *and* $b_0, b_1, \ldots, b_{m_p-1} \in S^0(\mathbb{R}^q \times \mathbb{R}^q)$. *Then the operator*

$$H_\pi^{s+\Re p+\delta-1/2}(\mathbb{R}^q; \mathbb{C}^{m_p}) \to H_P^{s,\delta}(\overline{\mathbb{R}}_+^{1+q}), \quad w \mapsto \sum_{0\le k<m_p} \Gamma_{pk}\left(b_k(y, D_y)w_k\right),$$

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$$f(y,\eta,t)c = \langle\eta\rangle^p \begin{pmatrix} \varphi(t)t^{-p} \\ -\varphi(t)t^{-p}\log t \\ \vdots \\ \frac{(-1)^{m_p-1}}{(m_p-1)!}\varphi(t)t^{-p}\log^{m_p-1} t \end{pmatrix} \cdot \pi(\eta)^{-1} \begin{pmatrix} b_0(y,\eta)c_0 \\ b_1(y,\eta)c_1 \\ \vdots \\ b_{m_p-1}(y,\eta)c_{m_p-1} \end{pmatrix}.$$

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In particular, $f \in S^{\Re p}(\mathbb{R}^q \times \mathbb{R}^q; \mathbb{C}^{m_p}, \mathbb{C}) \hat{\otimes} \mathscr{S}_P(\overline{\mathbb{R}}_+)$ as required.

Danke schön.