Subelliptic pseudo-differential operators on compact Lie groups

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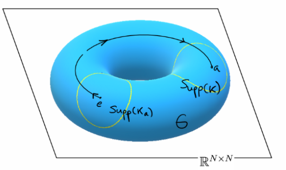
Outline

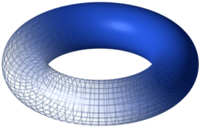
- Pseudo-differential calculus for sub-Riemannian structures on compact Lie groups. R.+Cardona, Subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups. 2020. (160pp). Using the coordinates-free approach with matrix-symbols as in my previous works with Turunen and Wirth.
- We introduce new Hörmander classes adapted to subelliptic operators, connecting the theory with the representation theory of the group, different to the classical approaches, e.g. Hörmander theory, Nagel and Stein classes[1] and many others.

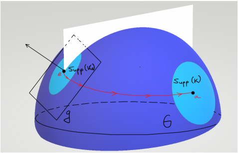
[1] Nagel, A. Stein, E. M. Some new classes of pseudodi erential operators. Harmonic analysis in Euclidean spaces, Part 2, pp. 159{169, Proc. Sympos. Pure Math., XXXV, Part I, Amer. Math. Soc.,Providence, R.I., 1979.

Our setting

- Lie groups = manifolds with symmetries.
- compact Lie groups = are diffeomorphic to closed subgroups of $\mathrm{U}(N) = fM\ 2\ \mathrm{C}^{N\ \ N} : M\ \ = M\ \ ^{1}g$ for N large enough.







- Examples: the torus $\mathrm{T}^n = (\mathrm{R}/\mathrm{Z})^n$, Linear Lie groups (groups of matrices), $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, etc. In particular, $\mathrm{SU}(2) = \mathrm{S}^3$.
- If M is a closed, connected and simply connected, then $M = \mathrm{S}^3$ (the Poincaré conjecture proved by Perelman). Our approach (with Turunen and Wirth, and with Cardona) induces global pseudo-differential theories on M.

Some preliminary information

- Analysis on compact Lie groups. Pseudo-differential calculus for Riemannian structures. R.+Turunen, Pseudo-differential operators and symmetries, Birkhäuser, 2010.
  With further developments with many co-authors (Turunen, Wirth, Dasgupta, Garetto, Akylzhanov, Tikhonov, Cardona, Kumar, ...).
- Analysis on compact Lie groups. Pseudo-differential calculus for sub-Riemannian structures. R.+Cardona, Subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups. 2020. arXiv:2008.09651 (160pp).
  With further developments with many co-authors (Cardona, Delgado, Federico, Kumar, ...).
- This approach to pseudo-differential operators on Lie groups may seem non-familiar for the $\mathbb{R}^n$-analysts since it relies on the representation theory of Lie groups; however, the representation theory that we use is quite simple, is very relevant, it clarifies/simplifies things, and it allows to attack global/local problems (e.g. global hypoellipticity, Fefferman L^p-bounds, Gårding type inequalities, Wodzicki residues, Index theory.);

Overview of global quantization theories

- Analysis on compact Lie groups. R.+Turunen, Pseudo-differential operators and symmetries, Birkhäuser, 2010.
  With further developments with many co-authors (Turunen, Wirth, Dasgupta, Garetto, Tikhonov, Delgado, Cardona, Kumar, ...).
- Analysis on nilpotent Lie groups. Open access book: Fischer+R., Quantization on nilpotent Lie groups, Birkhäuser, Progress in Math., 2016. With further developments with many co-authors (Fischer, Cardona, Yessirkegenov, Tokmagambetov, Rottensteiner,...).
- Analysis on locally compact type I groups.
  Mantoiu+R., Pseudo-differential operators, Wigner transform and Weyl systems on type I locally compact groups, Doc. Math. 2017.
- Global quantization on compact manifolds.
  R.+Delgado, J. d'Analyse Math, 2018.
- Nonharmonic analysis (of boundary value problems).
  R.+Tokmagambetov, IMRN, 2016.
- Global analysis on quantum groups, locally compact groups.
  R.+Akylzhanov, JFA 2020, +Majid CMP 2018.

Short overview

Pseudo-differential operators on $\mathbb{R}^n$ [Kohn+Nirenberg 1965, Hörmander 1967]:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x\cdot\xi}\, dx, \quad Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, \sigma_A(x,\xi)\, \widehat{f}(\xi)\, d\xi,$$

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \le C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|}, \ \langle\xi\rangle = (1+|\xi|^2)^{1/2}, \ \xi \in \mathbb{R}^n.$$

Ψ**DOs on the torus** $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$: Fourier coefficients with $\xi \in \mathbb{Z}^n$,

$$\widehat{f}(\xi) = \int_{\mathbb{T}^n} f(x)\, e^{-i2\pi x\cdot\xi}\, dx, \quad Af(x) = \sum_{\xi\in\mathbb{Z}^n} e^{i2\pi x\cdot\xi}\, \sigma_A(x,\xi)\, \widehat{f}(\xi),$$

$$\left|\Delta_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \le C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|}, \ \xi \in \mathbb{Z}^n.$$

[Agranovich 1990], [McLean 1991], [Turunen 2000], [R.+Turunen, JFAA, 2010].

Ψ**DOs on a compact Lie group** G: [R.+Turunen, Birkhaüser book, 2010]

$$\widehat{f}(\xi) = \int_G f(x)\, \xi(x)^*\, dx, \ Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\left(\xi(x)\, \sigma_A(x,\xi)\, \widehat{f}(\xi)\right),$$

$$\|\Delta_\xi^\alpha X^\beta \sigma_A(x,\xi)\|_{op} \le C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|}, \ \xi \in \widehat{G}, \ \langle\xi\rangle = e.v., \ \Delta_\xi = \text{diff. op.},$$

Basics of the representation theory

Unitary representation ξ of a group G is $\xi : G \to \mathcal{L}(H_\xi)$, where H_ξ is a Hilbert (representation) space, such that $\xi(x)^* = \xi(x)^{-1}$ (unitary) and $\xi(xy) = \xi(x)\,\xi(y)$ (preserves group structure).

It is irreducible if $\xi \neq \xi_1 \oplus \xi_2$ for some unitary representations ξ_1, ξ_2.

If G is compact, it is enough to consider finite-dimensional H_ξ, i.e. $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$ for $d_\xi = \dim H_\xi$ is the dimension of ξ.
Example: For $\mathbb{T}^n$, $\xi_k(x) = e^{2\pi i x\cdot k}$, $k \in \mathbb{Z}^n$. Then $\xi_k : \mathbb{T}^n \to \mathbb{C}^{1\times 1}$, $d_{\xi_k} = 1$.

The unitary dual $\widehat{G}$ is defined (omitting equivalent classes) as

$\widehat{G} = \{$ continuous irreducible unitary representations of $G\}$

Fourier coefficient $\widehat{f}(\xi)$ of $f \in L^1(G)$ at $\xi \in \widehat{G}$ is

$$\widehat{f}(\xi) = \int_G f(x)\,\xi(x)^*\,dx.$$

Note that $\widehat{f}(\xi) \in \mathcal{L}(H_\xi)$ is now an operator; a matrix if G is compact.

Fourier coefficient $\widehat{f}(\xi)$ of $f \in C^{\infty}(G)$ at $\xi \in \widehat{G}$ is

$$\widehat{f}(\xi) = \int_G f(x)\, \xi(x)^* \, dx.$$

Note that $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ is now a matrix! Fourier series is

$$f(x) = \mathcal{F}^{-1}(\widehat{f}) = \sum_{\xi \in \widehat{G}} d_\xi \operatorname{Tr}\left(\widehat{f}(\xi)\, \xi(x)\right).$$

<u>**Parseval's identity**</u> Let $f, g \in L^2(G)$. Then we have

$$(f, g)_{L^2(G)} = \sum_{\xi \in \widehat{G}} d_\xi \operatorname{Tr}\left(\widehat{f}(\xi)\, \widehat{g}(\xi)^*\right) =: (\widehat{f}(\xi), \widehat{g}(\xi))_{\ell^2(\widehat{G})},$$

so that $\|\widehat{f}\|^2_{\ell^2(\widehat{G})} := \sum_{\xi \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|^2_{HS}$ where $\|B\|_{HS} = \sqrt{\operatorname{Tr}(B B^*)}$.

Weight $h\xi i$ (Kumano-go notation)

We define the weight $h\xi i$ to measure the growth of the Fourier coefficients. This is essential if talking about function spaces and about the symbol classes associated to the unique bi-invariant Riemannian structure on G (Laplacian).

For each $\xi \; 2 \; \widehat{G}$, we have for the Laplacian:

$$\boxed{L_G \xi = \lambda^2 \xi}$$

In other words,

$$L_G \xi_{ij}(x) = \lambda^2 \xi_{ij}(x), \quad 1 \quad i, j \quad d \, .$$

We define

$$\boxed{h\xi i := (1 + \lambda^2)^{1=2}}$$

These are the eigenvalues of the first order elliptic operator $(1 \quad L_G)^{1=2}$, and we note that $h\xi i$ has here multiplicity d^2.

Matrix-valued subelliptic weight $\widehat{M}(\xi)$

- Consider the sub-Laplacian $L = \quad X_1^2 \qquad\qquad X_k^2,$ where $X = fX_1, \quad , X_k g$ satisfies the Hörmander condition (HC) at step κ.
- (HC) means the following. Let

$$X_I := [X_{i_1}, [X_{i_2}, \quad [X_{i_{\omega} \; 1}, X_{i_\omega}] \quad]], I = (i_1, i_2, \quad , i_!) \; 2 \; \mathbb{N}_0^!, \omega \; 2 \; \mathbb{N}.$$

Define the linear sub-space $H^! G := \text{span} f X_I : jIj \; 6 \; \omega g.$ Then X satisfies (HC) at step κ, if there exists $\kappa^0 \; 2 \; \mathbb{N}$ such that $H^{\;0} G = \mathfrak{g}$. We consider the smallest κ^0 with this property and we denote it by κ.

- The sub-Laplacian $L = \quad \sum_{j=1}^{k} X_j^2$ has L^2-discrete spectrum $f\nu_{ii}(\xi)^2 : 1 \quad i \quad d \; , [\xi] \; 2 \; \widehat{G} g$ and on every representation space it satisfies that

$$L\xi_{ij} \quad \widehat{L}(\xi)_{ij} := \nu_{ii}^2(\xi)\delta_{ij}, \, 1 \quad i, j \quad d$$

- Definition: (Matrix-valued subelliptic weight)

$$M(\xi) := (1 + \widehat{L}(\xi))^{\frac{1}{2}}, \;\; \widehat{L}(\xi) := \text{diag}[\nu_{ii}^2(\xi)]$$

Sobolev spaces, Besov spaces, Triebel-Lizorkin spaces

$f \in H^s(G) \iff \sum_{\xi\in\widehat{G}} d_\xi \langle\xi\rangle^{2s}\|\widehat{f}(\xi)\|_{\mathrm{HS}}^2 < \infty$.

$f \in C^\infty(G) \iff \forall M: \quad \|\widehat{f}(\xi)\| \leq C_M\langle\xi\rangle^{-M}$, any matrix norm.

$f \in \mathcal{D}'(G) \iff \exists M, C: \quad \|\widehat{f}(\xi)\| \leq C\langle\xi\rangle^{M}$, any matrix norm.

Besov, Triebel-Lizorkin spaces: R.+Nursultanov+Tikhonov (Ann. Sc. Norm. Pisa 2016).

Let $\eta \in C_0^\infty(\mathbb{R}^+,[0,1])$, $\eta \neq 0$, be such that $\mathrm{supp}(\eta) \subset [1/2,2]$, and that $\sum_{j\in\mathbb{Z}}\eta(2^{-j}\lambda)=1, \ \lambda>0$. holds. Fixing $\psi_0(\lambda) := \sum_{j=-\infty}^{0}\eta_j(\lambda)$, and for $j \geq 1$, $\psi_j(\lambda) := \eta(2^{-j}\lambda)$, we have

$$\sum_{\ell=0}^{\infty}\psi_\ell(\lambda)=1,\ \lambda>0.$$

Define $\psi_j(B)$ using the functional calculus of $B := (1+\mathcal{L}_G)^{\frac{1}{2}}$. Then, for $0<q<\infty$, and $1<p<\infty$, the Triebel-Lizorkin space $F^r_{p,q}(G)$ consists of the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{F^r_{p,q}(G)} := \left\| \left(\sum_{\ell=0}^{\infty} 2^{\ell r q} |\psi_\ell(B) f|^q \right)^{\frac{1}{q}} \right\|_{L^p(G)} < \infty.$$

Analysis of Weak-Triebel-Lizorkin spaces: R.+Cardona (Collect. Math., 2022).

The weak-$F^r_{1,q}(G)$ space is defined by the distributions $f \in \mathcal{D}'(G)$ such that

$$\|f\|_{\text{weak-}F^r_{1,q}(G)} := \sup_{t>0} t \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} 2^{\ell r q} |\psi_\ell(B) f(x)|^q \right)^{\frac{1}{q}} > t \right\} \right| < \infty . \quad (1)$$

Above, for a measurable subset $A \subset G$, $|A|$ denotes its Haar measure.

Embedding properties for $F^r_{p,q}(G)$ − spaces

Let G be a compact Lie group. Then we have the following properties:

(1) $F^{r+\varepsilon,L}_{p,q_1}(G) \hookrightarrow F^{r,L}_{p,q_1}(G) \hookrightarrow F^{r,L}_{p,q_2}(G) \hookrightarrow F^{r,L}_{p,\infty}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $0 \leq q_1 \leq q_2 \leq \infty$.

(2) $F^{r+\varepsilon,L}_{p,q_1}(G) \hookrightarrow F^{r,L}_{p,q_2}(G)$, $\varepsilon > 0$, $0 \leq p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.

(3) $F^r_{p,2}(G) = L^p_r(G)$ for all $r \in \mathbb{R}$, and all $1 < p < \infty$, where $L^p_r(G)$ are the standard Sobolev spaces on G.

Fourier multipliers on compact Lie groups

General pseudo-differential operators on compact Lie groups have been developed in R.+Turunen, Pseudo-differential operators and symmetries, Birkhäuser, 2010.
Fourier multipliers are special cases of this:

For $\sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ the corresponding Fourier multiplier is

$$\widehat{Af}(\xi) = \sigma(\xi)\widehat{f}(\xi)$$

Hence also, by the Fourier inversion formula above,

$$Af(x) = \sum_{\xi \in \widehat{G}} d_\xi \ \mathrm{Tr}(\xi(x)\sigma(\xi)\widehat{f}(\xi)).$$

We can also write A as the convolution operator: $Af = f * k, \ \sigma(\xi) = \widehat{k}(\xi).$

- Any left-invariant continuous linear operator on $C^\infty(G)$ is a Fourier multipliers. Many examples: the Laplacian L_G, any sub-Laplacian $L = -\sum_j X_j^2$, and their spectral multipliers.
- The Laplacian and and any sub-Laplacian are the key operators when defining pseudo-differential calculus for the Riemannian and sub-Riemannian structures, respectively.

Operators on compact Lie groups

If $A : C^\infty(G) \to \mathcal{D}'(G)$ cont. & linear, then we can define its symbol as

$$(x,\xi) \mapsto \sigma_A(x,\xi), \quad \sigma_A(x,\xi) := \xi(x)^* \, (A\xi)(x),$$

where $(A\xi)_{ij} = A(\xi_{ij})$ acts on components. This (full) symbol of A is matrix-valued: $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. Then we can show that

$$Af(x) = \sum_{\xi \in \widehat{G}} d_\xi \, \mathrm{Tr}\left(\xi(x)\, \sigma_A(x,\xi)\, \widehat{f}(\xi)\right)$$

This symbol is well-defined on $G \times \widehat{G}$ (non-commutative phase space). There are many familiar features, e.g. if

$$Af(x) = \int_G K(x,y)\, f(y)\, dy = \int_G f(y)\, R_A(x, y^{-1}x)\, dy,$$

then $\sigma_A(x,\xi) = \int_G R_A(x,y)\, \xi(y)^*\, dy$ (i.e. symbol is F.T. of the kernel).
We have (full) symbolic calculus for this quantization, and(!) with formulae resembling the familiar formulae on $\mathbb{R}^n$.

Difference operators on $\widehat{G}$

DEF A *di erence operator* (acting on Fourier coefficients) is

$$(\Delta_q \widehat{f})(\xi) := \widehat{(qf)}(\xi)$$

for $q \ 2 \ C^{1}(G)$ with $q(e) = 0$. Examples:

$\mathbb{R}^n$: $\frac{@}{@ \ j} = \Delta \quad ix_j$ since $\frac{@}{@ \ j}\widehat{f}(\xi) = (\quad \widehat{ix_j f})(\xi)$; $\quad ix_j j_{x=0} = 0$.

$\mathbb{T}^n$: Forward difference operator $\Delta_j \widehat{f}(\xi) = \widehat{f}(\xi + e_j) \quad \widehat{f}(\xi)$ on $\mathbb{Z}^n \ ' \ \widehat{\mathbb{T}^n}$ is given by $\Delta_j = \Delta_{q_j}$ with $q_j(x) = e^{\ 2 \ ix_j} \quad 1$. Here $q_j(0) = 0, q \ 2 \ C^{1}(\mathbb{T}^n)$.

Symbols: Recall $\sigma_A(x, \xi) = \widehat{R_A}(x, \ )(\xi)$ and Calderón-Zygmund theory.

DEF A collection $4_{q_1}, \ldots, 4_{q_k}$ is called *strongly admissible*, if $8j$:

$q_j \ 2 \ C^{1}(G)$, $q_j(e) = 0, dq_j(e) \ 6= 0$, $\mathrm{rank}(dq_1(e), \ldots, dq_k(e)) = \dim G$.

$\bigcap_j fx \ 2 \ G : q_j(x) = 0g = feg$ (i.e. e is the only common zero of q_j's)

For example, $f\Delta_{q_j} g_{j=1}^n, q_j(x) = e^{\ 2\pi i x_j} \quad 1$, is strongly admissible on $\mathbb{Z}^n \ ' \ \widehat{\mathbb{T}^n}$.

We use multi-index notation $q^\alpha(x) = q_1(x)^{\alpha_1} \quad q_k(x)^{\alpha_k}$ and $4_\xi^\alpha = 4_{q_1}^{\alpha_1} \quad 4_{q_k}^{\alpha_k}$.

Hörmander's class $\Psi^m(G)$

As usual, $\Psi^m(G)$ is the Hörmander class of pseudo-differential operators on G, i.e. such that their localisations have symbols in $S^m_{1;0}(\mathrm{R}^n)$.

Let $h\xi i$ be the eigenvalue of $(I \quad L)^{1=2}$, L Laplacian, corresponding to ξ.

Now, we define the class of symbols $\sigma_A \; 2 \; S^m_{;}(G)$ by

$$k4 \quad X_x \, \sigma_A(x, \xi) k_{op} \quad C \quad h\xi i^{m \quad j \; j+ \; j \; j}$$

Independent of the choice of a strongly admissible collection for $\rho > \delta$.

Theorem (R.+Turunen IMRN 2013; R.+Turunen+Wirth JFAA 2014)

$A \; 2 \; \Psi^m(G)$ if and only if $\sigma_A \; 2 \; S^m_{1;0}(G)$.

This result gives a link between Hörmander's theory and our matrix symbols.

Mihlin-Hörmander multipliers on compact Lie groups

Theorem: Mihlin-Hörmander multipliers (R.+Wirth, Math. Z. , 2015)

Let $\widehat{Af}(\xi) = \sigma(\xi)\widehat{f}(\xi)$. Let { be the smallest even integer $> \frac{1}{2}\dim G$ and let $1 < p < 1$. Then operator $A : C^1(G)$! $D^0(G)$ is $L^p(G)$-bounded if

$$kD \quad \sigma(\xi)k_{op} \quad C \ \ h\xi i \quad ^{j \ \ j} \text{ for all } j\alpha j \quad \{ .$$

This result can be extended to $S^m_{;}$ classes. We just give several examples.

Examples on S^3:

$\rho = \frac{1}{2}$: sub-Laplacian $L_s = D_1^2 + D_2^2$ has parametrix from op $S^{\ \ 1}_{\frac{1}{2};0}(S^3)$.
Hence $kuk_{L^p(S^3)} \quad C_p kL_s uk_{W^{p,j\frac{1}{p} \ \ \frac{1}{2}j \ \ 1}(S^3)}$. Also, e.g. for the "heat" operator, $kuk_{L^p(S^3)} \quad k(D_3 \quad D_1^2 \quad D_2^2)uk_{W^{p,j\frac{1}{p} \ \ \frac{1}{2}j \ \ 1}(S^3)}$.

$\rho = 0$: $\partial_X + c$ is invertible iff ic $\not 2$ $\frac{1}{2}\mathbb{Z}$. Then $(\partial_X + c) \ \ ^1 \ 2$ op $S^0_{0;0}(S^3)$.
Hence $kfk_{L^p(S^3)} \quad C_p k(\partial_X + c)fk_{W^{p,2j\frac{1}{p} \ \ \frac{1}{2}j}(S^3)}$, for all $1 < p < 1$.

Mihlin-Hörmander multipliers on compact Lie groups

Theorem: subelliptic Mihlin-Hörmander multipliers (R.+Cardona, [arXiv:2008.09651, pages 60-66, Section 6]).

Let us assume that G is a compact Lie group of dimension $n = 2d$ or $n = 2d + 1$, and that d is odd. Let $a \in \Sigma(\widehat{G})$ be a symbol satisfying

$$\|\widehat{M}(\xi)^{\frac{\rho|\alpha|+\varkappa(\kappa-\rho)}{\kappa}} D\, a(\xi)\|_{\mathrm{op}} \leq C, \quad |\alpha| \leq \varkappa := d + 1.$$

Then $A = \mathrm{Op}(a)$ extends to an operator of weak type $(1, 1)$. Moreover, if the dimension of the group is $\dim(G) = 2d$ or $\dim(G) = 2d + 1$, and d is even, the conclusion on A is the same provided that

$$\|\widehat{M}(\xi)^{\frac{\rho|\alpha|+\varkappa(\kappa-\rho)}{\kappa}} D\, a(\xi)\|_{\mathrm{op}} \leq C, \quad |\alpha| \leq \varkappa := d + 2.$$

In both cases, if $\rho = \kappa$, A extends to a bounded operator on $L^p(G)$, while for $0 \leq \rho < \kappa$, A extends to a compact bounded operator on $L^p(G)$ for all $1 < p < \infty$.

Mihlin-Hörmander multipliers on compact Lie groups for Triebel-Lizorkin spaces

Theorem: Mihlin-Hörmander multipliers R.+Cardona (Collect. Math. 2022)

Let us consider a Fourier multiplier $A : C^{\infty}(G) \to L^2(G)$ satisfying the symbol condition

$$\|\sigma\|_{l.u.,\, L^2_s(\widehat{G})} := \|\sigma\|_{L^{\infty}(\widehat{G})} + \sup_{r>0} r^{(s-\frac{n}{2})} \|\sigma\, \eta(r^{-1}\langle\xi\rangle)\|_{\dot{L}^2_s(\widehat{G})} < \infty, \qquad (2)$$

with $s > \frac{n}{2}$. Then $A \equiv T_\sigma$ extends to a bounded operator from $F^r_{p,q}(G)$ into $F^r_{p,q}(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$. For $p = 1$, A admits a bounded extension from $F^r_{1,q}(G)$ into weak-$F^r_{1,q}(G)$. Moreover

$$\|A\|_{\mathscr{B}(F^r_{p,q}(G))},\ \|A\|_{\mathscr{B}\left(F^r_{1,q}(G),\, \text{weak-}F^r_{1,q}(G)\right)} \lesssim \|\sigma\|_{l.u.,\, L^2_s(\widehat{G})}. \qquad (3)$$

The norm on the homogeneous Sobolev space $\dot{L}^2_s(\widehat{G})$ is given by $\|\sigma\|_{\dot{L}^2_s(\widehat{G})} := \||\cdot|^s \mathscr{F}_G^{-1}\sigma\|_{L^2(G)}$.

Mihlin-Hörmander-Marcinkiewicz multipliers on compact Lie groups for Triebel-Lizorkin spaces

Theorem: Mihlin-Hörmander-Marcinkiewicz multipliers R.+Cardona (Collect. Math. 2022)

Let σ be a symbol satisfying

$$\|\Delta^{\alpha}\sigma(\xi)\|_{\mathrm{op}} \leqslant C_{\alpha}\langle\xi\rangle^{-|\alpha|},\ |\alpha| \leqslant \varkappa := \left[\frac{n}{2}\right]+1. \qquad (4)$$

Then $A \equiv T_{\sigma}$ extends to a bounded operator from $F^{r}_{p,q}(G)$ into $F^{r}_{p,q}(G)$ for all $1 < p, q < \infty$, and all $r \in \mathbb{R}$. For $p = 1$, A admits a bounded extension from $F^{r}_{1,q}(G)$ into weak-$F^{r}_{1,q}(G)$. Moreover

$$\|A\|_{\mathcal{B}(F^{r}_{p,q}(G))}, \|A\|_{\mathcal{B}\left(F^{r}_{1,q}(G);\,\text{weak-}F^{r}_{1,q}(G)\right)} \lesssim \max\{C_{\alpha} : |\alpha| \leqslant \varkappa\}. \qquad (5)$$

Oscillating multipliers on compact Lie groups

Extensions of classical estimates for oscillating Fourier multipliers (Fefferman, Acta Math. 24, 9–36, (1970), and Fefferman and Stein Acta Math., 129, 137-193, (1972)) to compact Lie groups.

Theorem: Oscillating multipliers R.+Cardona (arXiv:2201.12883, 2022).

Let $j \quad j$ be the Carnot-Carathéodory distance on G associated to L, and let Q be the Hausdorff dimension of G. Assume that

$$\sup_{[\ ]2\widehat{G}} k\widehat{M}(\xi)^{\frac{Q\theta}{2}}\sigma(\xi)k_{\mathrm{op}} < 1\,, \tag{6}$$

for some $0 \quad \theta < 1$. If additionally, K satisfies the kernel condition

$$[K]_{H_{1,\theta}} := \sup_{R>0}\frac{1}{jB(R)j}\ s_{jyj<R}\ s_{jxj \quad 2R^{1\quad\theta}}\ jK(xy\quad^{1}) \quad K(x)jdxdy < 1\,, \tag{7}$$

then $A = T \quad : L^{1}(G) \ ! \ BMO_{L}(G)$ is bounded.

Oscillating multipliers on compact Lie groups

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Theorem: Oscillating multipliers R.+Cardona (arXiv:2201.12883, 2022).

Let $|\cdot|$ be the Carnot-Carathéodory distance on G associated to $\mathcal{L}$, and let Q be the Hausdorff dimension of G. Assume that

$$\sup_{[\xi]\in\widehat{G}} \|\widehat{\mathcal{M}}(\xi)^{\frac{Q\theta}{2}}\sigma(\xi)\|_{\mathrm{op}} < \infty, \qquad (8)$$

for some $0 \leq \theta < 1$. If additionally, K satisfies the kernel condition

$$[K]_{H_{1,\theta}} := \sup_{R>0} \frac{1}{|B(R)|} \int_{|y|<R} \int_{|x|\geq 2R^{1-\theta}} |K(y^{-1}x) - K(x)|dxdy < \infty, \qquad (9)$$

then $A = T_\sigma$ is bounded from the subelliptic Hardy space $H^{1,\mathcal{L}}(G)$ into $L^1(G)$.

Boundedness of pseudo-differential operators on compact Lie groups

Extensions of sharp L^p-boundedness theorems for Hörmander classes on $\mathbb{R}^n$ (Fefferman, Israel J. Math., 14, 413–417, (1973)) to compact Lie groups.

Theorem: R.+Delgado, J. Inst. Math. Jussieu, 18 (2019), 531–559.

Let G be a compact Lie group of dimension n. Let $A \in \Psi^{-n(1-\rho)/2}_{\rho,\delta}(G \times \widehat{G})$, $0 \leq \delta < \rho \leq 1$. Then $A = T_\sigma$ extends to a bounded operator from $L^\infty(G)$ to $BMO(G)$, and we have

$$\|A\|_{\mathcal{B}(L^\infty(G),BMO(G))} \leq C \max\{\|\sigma\|_{\ell, S^{-\frac{n(1-\rho)}{2}}_{\rho,\delta}}, \|\sigma_{A(1+\mathcal{L}_G)^{\frac{n(1-\rho)}{2}}}\|_{\ell, S^0_{\rho,\delta}}\}, \tag{10}$$

for $\ell \in \mathbb{N}$ large enough. Moreover, A also extends to a bounded operator from the Hardy space $H^1(G)$ into $L^1(G)$.

Boundedness of pseudo-differential operators on compact Lie groups

Extensions of sharp L^p-boundedness theorems for Hörmander classes on $\mathbb{R}^n$ (Fefferman, Israel J. Math., 14, 413–417, (1973)) to compact Lie groups.

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Let G be a compact Lie group of dimension n. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S^m_{\rho,\delta}(G \times \widehat{G})$, $0 \leq \delta < \rho \leq 1$. If $1 < p < \infty$, then $A = T_\sigma$ extends to a bounded operator from $L^p(G)$ to $L^p(G)$ provided that

$$m \leq m_p := n(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

From compact to nilpotent

There is now a number of applications of this theory on compact spaces:

Operator theory: Global functional calculus (R.+Wirth: JFA 2014);

Spectral theory: Grothendieck-Lidskii formulae, trace formulae for operators on L^p-spaces (Delgado+R., JMPA 2014, JFA 2014, MRL 2016), Gohberg formula (Dasgupta+R., JdAM, 2016);

PDEs: wave equations for sums of squares of vector fields, low regularity (Garetto+R., JDE 2015, ARMA 2015); sharp Gårding inequality (R.+Turunen, JFA 2011);

Harmonic analysis: Fourier Beurling algebras (Nursultanov+Tikhonov+R., SNS Pisa 2016), L^p-L^q Fourier multipliers (Akylzhanov+R., Studia Math 2016);

Function spaces: Dasgupta+R. (BSM 2014, TAMS 2016).

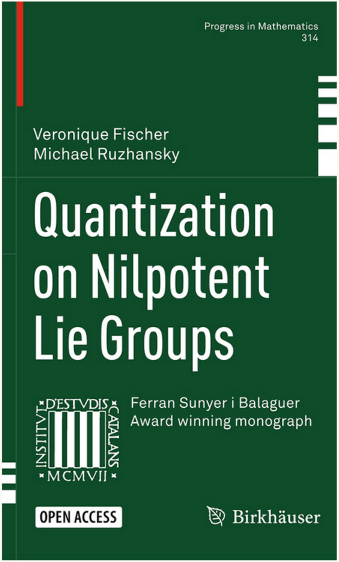
Homogeneous spaces: Extensions of above (starting with R.+Turunen, IMRN 2013);

Some of these techniques are lost once we move to nilpotent Lie groups: for example, there is no $\langle\xi\rangle$ to measure the growth of Fourier coefficients/symbols.

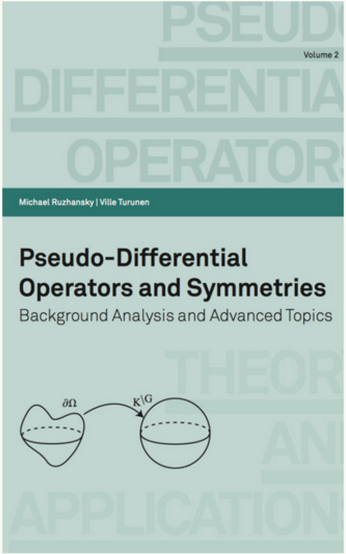
Important notational change: representations will be denoted by π instead of ξ to bring it more (notationally) into the realm of representation theory:
e.g. for an invariant differential operator D, $\sigma_D(\pi)$ $\pi(D)$ can be thought of as its infinitesimal representation, ...

A small advert: Pseudo-differential operators on Lie groups

2016, nilpotent Lie groups, open access



2010, compact Lie groups



Nilpotent Lie groups (joint work Fischer+R.)

Nilpotent Lie groups appear naturally in applications since many problems have nilpotent structure (sometimes hidden, e.g. Grushin operator, or e.g. Weyl quantization on $\mathbb{R}^n$, or modulation spaces) or can be approximated by a nilpotent structure.

Some literature: 70's: Stein+Rotschild, Folland, Dynin, Helffer+Nourrigat; 80's-90's: Beals, Greiner, Taylor, Geller, Hulanicki; Christ+et al., Melin, Strichartz, Shubin, ..., Ponge (MAMS 2008), Bahouri+Fermanian-Kammerer+Gallagher (Asterisque 2012);

Our setting: Heisenberg group; stratified and homogeneous Lie groups.
Natural class to be able to also use (hypoelliptic) differential operators are graded Lie groups; (in fact, this is usually the working assumption for nilpotent groups).
Important tool: Rockland operators.

Rockland: homogeneous differential operators for which infinitesimal representations (i.e. symbols) are injective on smooth vectors;
Beals; Helffer+Nourrigat's resolution of Rockland's conjecture: these are the homogeneous hypoelliptic differential operators.
Miller, Robinson+ter Elst: existence of Rockland operators ( ) group is graded.

Heisenberg group $\mathbb{H} \cong \mathbb{R}^3$

The law: $(x,y,t)(x',y',t') = (x+x', y+y', t+t'+\frac{1}{2}(xy' - x'y))$.
Lie algebra $\mathfrak{h}$: the basis X, Y, T is given by left-invariant vector fields

$$X = \partial_x - \frac{y}{2}\partial_t, \quad Y = \partial_y + \frac{x}{2}\partial_t, \quad \text{and} \quad T = \partial_t.$$

Canonical commutation relations:

$$[X, Y] = T, \qquad \text{and } T \text{ commutes with everything.}$$

Stratification/gradation: $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1 = \mathbb{R}X \oplus \mathbb{R}Y$, $\mathfrak{g}_2 = \mathbb{R}T$.

Dilations D_r:

$$D_r(x,y,t) = r(x,y,t) = (rx, ry, r^2 t), \quad (x,y,t) \in \mathbb{H},\ r > 0.$$

Positive Rockland operator $R = -L$ with the sub-Laplacian $L = X^2 + Y^2$.
It is positive, hypoelliptic, homogeneous (of degree 2):
$R(f \circ D_r) = r^2 (Rf) \circ D_r$.

Graded groups extend this construction

The law: G is a nilpotent connected simply connected Lie group (actually, it can be always realised as some $\mathbb{R}^n$ with a polynomial group law).

Lie algebra $\mathfrak{g}$: the basis $X_1, \dots, X_n$ of left-invariant vector fields.
Commutation relations: any, but 'nilpotent' implies finitely many non-zero.

Gradation: $\mathfrak{g} = \bigoplus_{\ell=1}^{\infty} \mathfrak{g}_\ell$ with $[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'}$, where the $\mathfrak{g}_\ell$ are almost all equal to $\{0\}$. $\mathfrak{g}$ is called stratified if $\mathfrak{g}_1$ generates $\mathfrak{g}$ in this way (Ex: $\mathbb{R}^n, \mathbb{H}$).
Exponential mapping $\exp_G : \mathfrak{g} \to G$ is a diffeo and identifies points $(x_1, \dots, x_n) \in \mathfrak{g} \simeq \mathbb{R}^n$ with points $x = \exp_G(x_1 X_1 + \cdots + x_n X_n) \in G$.
Lebesgue measure on $\mathfrak{g}$ is the Haar measure on G. $\mathcal{S}(G) := \mathcal{S}(\mathfrak{g}) \circ \exp_G^{-1}$.
Dilations $D_r : \mathfrak{g} \to \mathfrak{g}$ are given by $D_r X = r^\ell X$ for every $X \in \mathfrak{g}_\ell$.
The weights of the dilations are $\upsilon_1, \dots, \upsilon_n \in \mathbb{N}$ given by $D_r X_j = r^{\upsilon_j} X_j$, $j = 1, \dots, n$. The associated group dilations are defined by

$$r \cdot x := (r^{\upsilon_1} x_1, r^{\upsilon_2} x_2, \dots, r^{\upsilon_n} x_n), \quad x = (x_1, \dots, x_n) \in G, \ r > 0.$$

Positive Rockland operator $\mathcal{R}$ is a positive, hypoelliptic, homogeneous (of degree ν) operator: $\mathcal{R}(f \circ D_r) = r^\nu (\mathcal{R} f) \circ D_r$. It always exists, but $\nu \geq 2$.

Difficulties (in the nilpotent case) compared to compact

Representations $\pi : G \to U(H)$ of nilpotent Lie groups are usually infinite-dimensional (unless $G = \mathbb{R}^n$): π is irreducible but $\dim H = \infty$.

Therefore, symbols $\sigma(x,\pi) : H \to H$ are no longer matrices but operators.

Usually, the symbols $\sigma(x,\pi)$ do not have to be bounded on H (for example, for differential operators). However, they can be made (good) sense of as (densely defined) operators $\sigma(x,\pi) : H^\infty \to H$ acting on smooth vectors H^∞ of π. In fact, they satisfy $\sigma(x,\pi) : H^\infty \to H^\infty$ and hence are composable. We always have the quantization

$$Tf(x) = \int_{\widehat{G}} \mathrm{Tr}\,(\pi(x)\sigma(x,\pi)\widehat{f}(\pi))d\mu(\pi),$$

for the Plancherel measure $d\mu(\pi)$, $\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx$.

Example of $\mathbb{H} \simeq \mathbb{R}^3$: $d\mu(\pi) = |\lambda| d\lambda$, $\lambda \in \mathbb{R}\setminus 0$, for Schrödinger representations $\pi : \mathbb{H} \to U(L^2(\mathbb{R}))$, $H_\lambda = L^2(\mathbb{R})$, $H^\infty_\lambda = S(\mathbb{R})$. For sub-Laplacian $\sigma_L(x,\pi) = |\lambda|(\partial_u^2 - u^2)$ is the harmonic oscillator on $\mathbb{R}$.

Quantization on graded Lie groups

In our book, Fischer+R., Quantization on nilpotent Lie groups, Progress in Math., Birkhäuser, 2016, 550 pages, we

review the theory of graded and homogeneous Lie groups, homogeneous operators (of complex degrees, extending Folland-Stein or Ricci in this respect) and their L^p-L^q bounds, and associated Rockland operators;

develop the theory of Sobolev spaces $L^p_s, 1 \quad p \quad 1$, on graded Lie groups (using the heat semigroup associated to a positive Rockland operator): embeddings, interpolation, properties of Bessel and Riesz kernels, etc. This extends the theory of Sobolev spaces on stratified groups as developed by Folland. Ultimately, we show that Sobolev spaces do not depend on the choice of a particular positive Rockland operator: hence, in particular, we can also use higher degree Rockland operators also on the Heisenberg and on stratified groups.

Quantization on graded Lie groups

We introduce symbol classes $S^{m}_{;}(G)$ associated to a positive Rockland operator. This is technically more involved than in the case of compact Lie groups because symbols are operator-valued and because the Rockland operator is not central. However, one is greatly assisted by the underlying homogeneous structure, especially when developing the analysis of difference operators. We show that these symbol classes actually do not depend on the choice of a positive Rockland operator used to define them.

Symbols $\sigma(x,\pi) : H^{1} \ ! \ H^{1}$ act on smooth vectors $H^{1} \quad H$ of π, smooth in x. We say that $\sigma \ 2 \ S^{m}_{;}(G)$ if

$$k\pi(I+R)^{\frac{m+\rho[\alpha] \quad \delta[\beta]+\gamma}{\nu}} \Delta \ X \ \ \sigma(x,\pi)\pi(I+R)^{\ \frac{\gamma}{\nu}} k_{L(H_{\pi})} \quad C$$

holds for all α, β, γ. Here $[\alpha] = \sum_j \upsilon_j \alpha_j$, $\Delta \ \ \widehat{\kappa}(\pi) = \widehat{x \ \ \kappa}(\pi)$. Note:

$\star$ actually, it is enough (equivalent) to assume this for one fixed γ (e.g. for $\gamma = 0$, R.+Cardona 2020)
$\star$ in compact case $\pi(I+R)^{1=}$ is scalar $= h\pi i$ and can be taken out.

Properties of symbol classes

In different classes (e.g. operators in $L(L^2(G))$ and Hilbert-Schmidt symbols) there is 1-1 correspondence between symbols and operators through

$$T\ f(x) = \int_{\widehat{G}} \mathrm{Tr}\ (\pi(x)\sigma(x,\pi)\widehat{f}(\pi))d\mu(\pi).$$

The operators in the classes $\mathrm{Op}(S^{m}_{;}(G))$ are more singular than their Euclidean counterparts. However, they fall within the Calderón-Zygmund theory on spaces of homogeneous type as developed by Coifman and Weiss.

In the case of $G = \mathrm{H}$ the Heisenberg group, the symbol classes $S^{m}_{;}(\mathrm{H})$ can be characterised by their λ-symbols a via the Weyl quantization. Namely, their λ-symbols are in the λ-dependent Shubin classes (a semiclassical version of Shubin classes based on the harmonic oscillator), with an essential dependence on λ (the derivative ∂ is involved in their definition). λ-symbol classes are uniform within the Weyl-Hörmander theory.

Selection of results (on graded Lie groups)

Symbol classes $\cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ form an algebra stable under taking transpose or adjoint. We can write $\Psi^m_{\rho,\delta}(G) := \mathrm{Op}(S^m_{\rho,\delta}(G))$.

$\rho \neq 0$. Operators in $\cup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}$ form an algebra stable under transpose or adjoint, with asymptotic formulae for the symbolic calculus (for $\rho > \delta$).

Ellipticity, hypoellipticity, parametrices in these classes.

The set of operators $\cup_{m\in\mathbb{R}} \Psi^m_{\rho,\delta}$ contains the left-invariant calculus.

The kernels are of Calderón-Zygmund type on homogeneous Lie groups.

Calderón-Vaillancourt: If $\rho \in [0,1)$, operators $\Psi^0_{\rho,\rho}$ are bounded on $L^2(G)$.

Calderón-Zygmund theory $\Longrightarrow$ boundedness in $L^p(G)$ for zero order operators. Calculus $\Longrightarrow$ properties in Sobolev spaces $L^p_s(G)$. Results for Fourier multipliers.

Results for spectral multipliers, space-dependent spectral multipliers, spectral multipliers in a finite collection of strongly commuting Rockland operators.

$(1+R)^{\frac{m}{\nu}} \in \Psi^m_{1,0}$, where R is a positive Rockland operator of degree ν. Moreover, functional calculus for symbols and operators in these classes.

Positive operators satisfy sharp Gårding inequalities (Fischer+R., CRAS 2013).

New: Sobolev spaces on $\widehat{G}$, Mihlin-Hörmander theorem on graded Lie groups.

Examples on the Heisenberg group: for sub-Laplacian L, (Fischer+R., CRAS 2014)

$a(x)L$ satisfies sharp Gårding inequality;

L is Schwartz hypoelliptic: $f \in S'(G)$, $(1-L)f \in S(G) \Longrightarrow f \in S(G)$.

Boundedness of pseudo-differential operators on graded Lie groups

Extensions of sharp L^p-boundedness theorems for Hörmander classes on $\mathbb{R}^n$ (Fefferman, Israel J. Math., 14, 413–417, (1973)) to graded Lie groups.

Theorem: R.+Cardona+Delgado (J. Geom. Anal., 2021)

Let G be a graded Lie group of homogeneous dimension Q. Let $A \in \Psi^{-\frac{Q(1-\rho)}{2}}_{\rho,\delta}(G \times \widehat{G})$, $0 \le \delta \le \rho \le 1$, $\delta \neq 1$. Then $A = \mathrm{Op}(\sigma)$ extends to a bounded operator from $L^1(G)$ to $BMO(G)$, and we have

$$\|A\|_{\mathscr{B}(L^1(G);BMO(G))} \le C \max\{\|\sigma\|_{\ell; S^{-\frac{Q(1-\rho)}{2}}_{\rho,\delta}}, \|\sigma \mathcal{A}(1+\mathcal{R})^{\frac{Q(1-\rho)}{2\nu}}\|_{\ell; S^0_{\rho,\delta}}\}, \quad (11)$$

for $\ell \in \mathbb{N}$ large enough. Moreover, A also extends to a bounded operator from the Hardy space $H^1(G)$ into $L^1(G)$.

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Let G be a graded Lie group of homogeneous dimension Q. Let $A : C^\infty(G) \to \mathcal{D}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S^m_{\rho,\delta}(G \times \widehat{G})$, $0 \le \delta \le \rho \le 1$, $\delta \neq 1$. If $1 < p < \infty$, then $A = T_\sigma$ extends to a bounded operator from $L^p(G)$ to $L^p(G)$ provided that

$$m \le m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Setting: G is a locally compact group, second countable, type I, and unimodular (for simplicity).

Type I: sometimes called postliminar, e.g. [Folland] has many formulations.

Examples: compact, abelian, Euclidean, real algebraic groups, exponential groups (in particular nilpotent), semi-simple Lie groups.

Important property: $\widehat{G}$ has the so-called Mackey Borel structure and has the Plancherel measure $d\mu(\xi)$. Symbols of e.g. $L^2(G)$-bounded invariant operators will belong to the von Neumann algebra $L(\widehat{G}) := \int_{\widehat{G}} L(H\ )d\mu(\xi)$.

Fourier transform: for $f\ 2\ L^1(G)$, $\widehat{f}(\pi) = \int_G f(x)\xi(x)\ dx\ 2\ L(H\ )$.
This extends to $L^2(G)$ with $f(x) = \int_{\widehat{G}} \mathrm{Tr}\,(\xi(x)\widehat{f}(\xi))d\mu(\xi)$.
The quantization formula is

$$(\mathrm{Op}(a)f)(x) = \int_{\widehat{G}} \mathrm{Tr}\,(\xi(x)a(x,\xi)\widehat{f}(\xi))d\mu(\xi).$$

Let us consider the sub-Laplacian $L = \quad \sum_{i=1}^{k} X_i^2$.

- Definition. Subelliptic Hörmander classes: Let $0 \quad \delta, \rho \quad 1$. Then, $a \; 2 \; S^{m;L}_{;}(G \quad \widehat{G})$ if it satisfies the following symbols estimates

$$kak_{j \;\; j+j \;\; j; S^{m,L}_{\rho,\delta}} := \sup_{(x;[\;])2G \quad \widehat{G}} k\widehat{M}(\xi)^{(\;j\;j\quad j\;j\quad m)} \partial_X \Delta \;\; a(x,\xi) k_{op} < 1 .$$

For the subelliptic pseudo-differential classes we have stability of the theory under compositions and adjoints. Define

$$\Psi^{m;L}_{;}(G \quad \widehat{G}) := fT \quad : \sigma \; 2 \; S^{m;L}_{;}(G \quad \widehat{G}) g,$$

for $0 \quad \delta \quad \rho \quad 1$. Then:

- If $T \quad 2 \; \Psi^{m_1;L}_{;}(G \quad \widehat{G})$ and $T \quad 2 \; \Psi^{m_2;L}_{;}(G \quad \widehat{G})$, then, $T \quad T \quad 2 \; T \quad 2 \; \Psi^{m_1+m_2;L}_{;}(G \quad \widehat{G})$,
- If $T \quad 2 \; \Psi^{m;L}_{;}(G \quad \widehat{G})$ then $T \quad 2 \; \Psi^{m;L}_{;}(G \quad \widehat{G})$.

Theorem: Singular kernels of subelliptic operators

Let $A \in \Psi^{m,L}_{\rho,\delta}(G \times \widehat{G})$. Then, the right-convolution kernel of A, $x \mapsto k_x : G \to C^\infty(G \setminus \{e_G\})$, defined by $k_x := \mathscr{F}^{-1}\sigma(x,\cdot)$, satisfies the following estimates for $|y| < 1$:

(i) if $m > -Q$, there exists $\ell \in \mathbb{N}$, independent of σ, such that

$$|k_x(y)| \lesssim_m \|\sigma\|_{\ell, S^{m,L}_{\rho,\delta}} |y|^{-\frac{Q+m}{\rho}}.$$

(ii) If $m = -Q$, there exists $\ell \in \mathbb{N}$, independent of σ, such that

$$|k_x(y)| \lesssim_m \|\sigma\|_{\ell, S^{m,L}_{\rho,\delta}} |\log|y||.$$

(iii) If $m < -Q$, there exists $\ell \in \mathbb{N}$, independent of σ, such that

$$|k_x(y)| \lesssim_m \|\sigma\|_{\ell, S^{m,L}_{\rho,\delta}}.$$

Asymptotic expansions

Theorem: Asymptotic expansion for the adjoint operator

Let 0 6 $\delta < \rho$ 6 1. If A 2 $\mathrm{Op}(S^{m;L}_{;}(G \quad \widehat{G}))$, then A 2 $\mathrm{Op}(S^{m;L}_{;}(G \quad \widehat{G}))$. The symbol of A , $\widehat{A}\ (x,\xi)$ satisfies the asymptotic expansion,

$$\widehat{A}\ (x,\xi) \quad \sum_{j \ j=0}^{1} \Delta \ \ \partial_X^{(\ )}(\widehat{A}(x,\xi) \ \ ). \qquad (12)$$

This means that, for every N 2 N, and all ℓ 2 N,

$$\Delta \ \ {}^{\ell}\partial_X^{(\ )} \left(\widehat{A}\ (x,\xi) \quad \sum_{j \ \ j 6 N} \Delta \ \ \partial_X^{(\ )}(\widehat{A}(x,\xi) \ \ ) \right) 2 \ S^{m \ \ (\ \ )(N+1) \ \ `+ \ j \ \ j;L}_{;},$$

where $j\alpha{\cdot}j = \ell$.

Asymptotic expansions

Theorem: Asymptotic expansion for the composition of two operators

Let $0 \ 6 \ \delta < \rho \ 6 \ 1$. If $A_i \ 2 \ \Psi^{m_i;L}_{;}(G \quad \widehat{G})$, $i = 1, 2$, then $A := A_1 \quad A_2 \ 2 \ \Psi^{m_1+m_2;L}_{;}(G \quad \widehat{G})$. The symbol of A, $\widehat{A}(x,\xi)$, satisfies the asymptotic expansion,

$$\widehat{A}(x,\xi) \quad \sum_{j \ j=0}^{1} (\Delta \ \widehat{A}_1(x,\xi))(\partial_X^{(\ )}\widehat{A}_2(x,\xi)), \tag{13}$$

this means that, for every $N \ 2$ N, and all $\ell \ 2$ N,

$$\Delta \ ^{\ell}\partial_X^{(\ )}\left(\widehat{A}(x,\xi) \quad \sum_{j \ j 6 N}(\Delta \ \widehat{A}_1(x,\xi))(\partial_X^{(\ )}\widehat{A}_2(x,\xi))\right)$$

$$2 \ S^{m_1+m_2 \ (\quad)(N+1) \quad `+ \ j \ j;L}_{;}(G \quad \widehat{G}),$$

for every $\alpha` \ 2 \ N_0^n$ with $j\alpha` j = \ell$.

Boundedness of subelliptic pseudo-differential operators on compact Lie groups

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Theorem: R.+Cardona, (2020).

Let G be a compact Lie group of Hausdorff dimension Q. Let $A \in \Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$. $0 \leq \delta < \rho \leq 1$. Then $A = T_\sigma$ extends to a bounded operator from $L^1(G)$ to $BMO(G)$, and we have

$$\|A\|_{\mathcal{B}(L^1(G),BMO_{\mathcal{L}}(G))} \leq C \max\{\|\sigma\|_{\ell, S_{\rho,\delta}^{-\frac{Q(1-\rho)}{2},\mathcal{L}}}, \|\sigma \mathcal{A}(1+\mathcal{R})^{\frac{Q(1-\rho)}{2\nu}}\|_{\ell, S^0_{\rho,\delta}}\}, \tag{14}$$

for $\ell \in \mathbb{N}$ large enough. Moreover, A also extends to a bounded operator from the subelliptic Hardy space $H^{1,\mathcal{L}}(G)$ into $L^1(G)$.

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Theorem: R.+Cardona, (2020).

Let G be a compact Lie group of Hausdorff dimension Q. Let $A : C^{\infty}(G) \to \mathcal{D}'(G)$ be a pseudo-differential operator with symbol $\sigma \in S^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$, $0 \leq \delta < \rho \leq 1$. If $1 < p < \infty$, then $A = T_\sigma$ extends to a bounded operator from $L^p(G)$ to $L^p(G)$ provided that

$$m \leq m_p := Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Other properties of the subellitic calculus

- (Calderón-Vaillancourt Theorem). $T \quad : L^2(G) \;\; ! \;\; L^2(G)$ is bounded if $m = 0$ and $1 \quad \delta \quad \rho \quad 1$, $\delta < 1/\kappa$.

- We prove a subelliptic version of the Hulanicki Theorem [2]
- The subelliptic calculus is stable under the spectral functional calculus of the sub-Laplacian:
  - Let $f \; 2 \; S^{\frac{m}{2}}(\mathrm{R}_0^+)$, $m \; 2 \; \mathrm{R}$, be such that $j\partial_\lambda^k f(\lambda)j \;.\; (1+\lambda)^{\frac{m}{2} \quad k}$. Then, for all $t > 0$,

$$f(tL) = \int_0^{1} f(t\lambda)dE_\lambda \; 2 \; S_{1,0}^{m,L}(G \quad \widehat{G}).$$

  - Example (stability under real powers):

$$8s \; 2 \; \mathrm{R}, \;\; (1+L)^{\frac{s}{2}} \; 2 \; \Psi_{1,0}^{s,L}(G \quad \widehat{G}).$$

[2] using the fundamental estimates in Jerison, D., Sanchez-Calle, A. Subelliptic second order di erential operators, in Complex analysis III, Springer, 1987, 46{77.

The subelliptic functional calculus on G.

- The subelliptic calculus is stable under the action of the complex functional calculus. It extends to the one for the elliptic Hörmander classes (R.+Wirth, J. Funct. Anal. 267, 144–172, (2014)).

$$F(A) := -\frac{1}{2\pi i} \oint_{\partial\Lambda_\varepsilon} F(z)(A - zI)^{-1} dz, \; A = T_a. \tag{15}$$

  - Let $m > 0$, and let $0 \leqslant \delta < \rho \leqslant 1$. Let $a \in S^{m,L}_{\rho,\delta}(G \times \widehat{G})$ be a parameter L-elliptic symbol with respect to Λ. Let us assume that F satisfies the estimate $|F(\lambda)| \leqslant C|\lambda|^s$ uniformly in λ, for some $s \in \mathbb{R}$.

Then $\sigma_{F(A)} \in S^{ms,L}_{\rho,\delta}(G \times \widehat{G})$, for F satisfying some suitable conditions (next slide).

Then $\sigma_{F(A)} \in S^{ms;L}_{;}(G \times \widehat{G})$, for F satisfying some suitable conditions. Namely:

(I). $\Lambda_{\varepsilon} := \Lambda \cup \{z : |z| \leq \varepsilon\}$, $\varepsilon > 0$, and $\Gamma = \partial\Lambda_{\varepsilon} \subset \mathrm{Resolv}(A)$ is a positively oriented curve in the complex plane $\mathbb{C}$.

(II). F is an holomorphic function in $\mathbb{C} \setminus \Lambda_{\varepsilon}$, and continuous on its closure.

(III). We will assume decay of F along $\partial\Lambda_{\varepsilon}$ in order that the operator (15) will be densely defined on $C^{\infty}(G)$ in the strong sense of the topology on $L^2(G)$.

- Example (stability under complex powers):

$$\forall z \in \mathbb{C},\ (1+\mathcal{L})^{\frac{z}{2}} \in \Psi^{\mathrm{Re}(z);\mathcal{L}}_{1;0}(G \times \widehat{G}).$$

Some results related to PDE

- (subelliptic Garding Inequality).

$$\mathrm{Re}(a(x,D)u,u) > C_1 kuk_{L^{2,L}_{\frac{m}{2}}(G)} \quad C_2 kuk^2_{L^2(G)}$$

- Well-posedness for the Cauchy problem (with $P(t,x,D)\ 2\ \Psi^{m;L}_{;}$ being strongly L-elliptic)

$$\frac{\partial v}{\partial t} = P(t,x,D)v + f,\ v(0) = u_0, v\ 2\ D^{0}((0,T) \quad G)$$

- Sharp Gårding inequality, (R+Cardona+Federico, 2021).

$$8a\ 2\ S^{m;L}_{;}, 8(x,[\xi]), a(x,[\xi]) \quad 0,\ 0 \quad \delta < (2\kappa \quad 1)\ ^{1}\rho,$$

$$\mathrm{Re}(a(x,D)u,u) > \quad C kuk^2_{L^{m}\ (\frac{\rho}{\kappa}\ \ (2\ \ \frac{1}{\kappa})\delta)(G)}.$$

- The subelliptic sharp Gårding inequality extends the one for the elliptic classes on compact Lie groups, see R+Turunen, J. Funct.Anal. 2011.

Some results related to Spectral theory

- Asymptotic expansions in spectral geometry

$$\mathbf{Tr}(A\psi(tE)) = t \quad ^{\frac{Q+m}{q}} \left(\sum_{k=0}^{1} a_k t^k\right) + \frac{c_Q}{q}\int_0^1 \psi(s) \quad \frac{ds}{s}.$$

- $A\ 2\ \Psi^{m;L}_{;}(G \quad \widehat{G})$, $0\ 6\ \delta,\rho\ 6\ 1$, that

$$\mathbf{Tr}(Ae \quad ^{t(1+L)^{\frac{q}{2}}}) \quad t \quad ^{\frac{m+Q}{q}} \sum_{k=0}^{1} a_k t^{\frac{k}{q}} \quad \frac{b_0}{q}\log(t), \quad t\ !\ 0^+, \qquad (16)$$

  for $m > \quad Q$. If $m = \quad Q$, then $a_k = 0$ for every k, and for $m > \quad Q$, $b_0 = 0$.

- We recover classical results. In the case of the elliptic classes, for $Q = n = \dim(G)$, and $m = \quad n$, $b_0 = \mathrm{res}(A)$ agrees with the Wodzicki residue of the operator.

Summarising the results for the subelliptic calculus

Symbol classes $[_{m2\mathrm{R}} S^{m,L}_{\rho,\delta}$ form an algebra stable under taking transpose or adjoint. We can write $\Psi^{m,L}_{\rho,\delta}(G) := \mathrm{Op}(S^{m,L}_{\rho,\delta}(G))$.

$\rho \,6\!\!= 0$. Operators in $[_{m2\mathrm{R}} \Psi^{m}_{\rho,\delta}$ form an algebra stable under transpose or adjoint, with asymptotic formulae for the symbolic calculus (for $\rho > \delta$).

Ellipticity, hypoellipticity, complex functional calculus, Hulanicki theorem, parametrices in these classes.

The kernels are of Calderón-Zygmund type in the sense of Coifman and Weiss.

Calderón-Vaillancourt: If $\rho\ 2\ [0,1)$, $0 \quad \delta \quad \rho$, $\delta < 1/\kappa$, operators in $\Psi^{0}_{\rho,\delta}$ are bounded on $L^2(G)$. More L^p-estimates include subelliptic versions of Fefferman and Fefferman and Stein type estimates. Results for spectral multipliers of sub-Laplacian. Our approach allows some extensions to Lie groups of pol. growth.

$(1+L)^{\frac{m}{2}}\ 2\ \Psi^{m,L}_{1,0}$. Moreover, real and complex functional calculus for symbols and operators in these classes. The subelliptic calculus is a parallel theory to the one in graded Lie groups (R+Fischer).

Positive operators satisfy sharp Gårding inequalities (R+Cardona+Federico, 2021).

New: Mihlin-Hörmander theorems for Triebel-Lizorkin spaces on compact Lie groups. New: Connecting Index theory, non-commutative geometry, Wodzicki residues with the representation theory of the group, (R.+Cardona+Delgado).

Thank you for your attention