Bergman-Szegő kernel asymptotics in weakly pseudoconvex finite type cases

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Riemann mapping theorem

Let $\mathbb{D}^n := \{z \in \mathbb{C}^n | \, |z| < 1\} \subset \mathbb{C}^n$.

Theorem (Riemann mapping theorem 1851)

Let $U \subset \mathbb{C}$ open, proper, simply connected. Then $U \cong \mathbb{D}^1$ (biholomorphic)

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Naive higher dimensional generalization is false

Theorem (Poincare 1907)

In $\mathbb{C}^2$, the polydisk and disk $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$ (not biholomorphic).

Poincare's proof computes:
Aut $(\mathbb{D}^n) = PSU(n, 1) :=$
$\left\{ T = \begin{bmatrix} A & b \\ c & d \end{bmatrix} | T \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} T^* = \begin{bmatrix} I_n & \\ & -1 \end{bmatrix}, \det T = 1 \right\} / S^1$ where $T.z = \frac{Az+b}{cz+d}$

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Questions:

-find generalization of RMT in higher dimensions?

-biholomorphism classification of domains in higher dimensions

-more robust biholomorphism invariants?

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$$L^2(U) := \left\{ f : U \to \mathbb{C} \,\text{measurable} \mid \|f\|^2_{L^2(U)} = \int_U |f|^2 \, d\mathrm{vol} < \infty \right\}$$

$$H^0_{(2)}(U) := \left\{ f \in L^2(U) \,|\bar{\partial} f = 0 \right\} \subset L^2(U)$$

Bergman projection: $\Pi_U : L^2(U) \to H^0_{(2)}(U) \subset L^2(U)$

Bergman kernel: $\Pi_U \in L^2(U \times U)$ satisfies

$$(\Pi_U f)(z) = \int_U \Pi_U(z, z') \, f(z') \, d\mathrm{vol}(z')$$

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Properties:
1. $\Pi_U(z, z') = \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(z')}$, for $\{\varphi_j\}_{j=1}^{\infty}$ ONB of $H^0_{(2)}(U)$.
2. $\overline{\Pi_U(z, z')} = \Pi_U(z', z)$
3. $\Pi_U(z, z')$ hol./antihol. in z/z', smooth in the interior
4. (U bounded) $\Pi_U(z, z) > 0$ in the interior

Bergman metric

The Bergman metric on U is defined via

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Theorem (Bergman 1931)

Let $F : U_1 \to U_2$ be a biholomorphism between two domains. Then it is an isometry $F_* g^{U_1} = g^{U_2}$.

Proof.

From defining property $\Pi_{U_1}\left(z,w\right) = \det\left(\frac{\partial F}{\partial z}\right)\left[\Pi_{U_2}\left(F\left(z\right),F\left(w\right)\right)\right]\overline{\det\left(\frac{\partial F}{\partial w}\right)}$.
Mixed partials of Jacobians are zero.

□

Computations of Π_U

1. $U = \mathbb{D}^n$ (disk)

ONB: $\sqrt{\frac{\binom{\alpha+n}{n}}{\pi^n}} z^\alpha$

$$\Pi_{\mathbb{D}^n}(z,z') = \frac{n!}{\pi^n} \frac{1}{\left(1 - z\overline{z'}\right)^{n+1}}$$

$g^{\mathbb{D}^n}$ corresponds to Poincare disk model for hyperbolic space

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2. (D'Angelo '78)
$U = E_p := \left\{ |z_1|^2 + |z_2|^{2p} \leq 1 \right\}$ (ellipsoid)

$$\Pi_{E_p}\left(z, z'\right) = \frac{2}{\pi^2} \frac{1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^3} + \frac{2}{\pi^2} \frac{p-1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^2}$$

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Computing curvatures: $\mathbb{D}^1 \times \mathbb{D}^1 \not\cong \mathbb{D}^2$, $E_p \not\cong E_{p'}$ for $p \neq p'$ (not biholomorphic).

Fefferman's theorem

Another strategy for $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$: look for behaviour of biholomorphisms near boundary (boundary of polydisk is non-smooth)
Conjecture: (cf. Krantz '13) Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded. Then any biholomorphism $F : U_1 \to U_2$ extends smoothly to the boundary.

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Theorem (Fefferman '74)

Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded and strongly pseudoconvex. Then $U_1 \cong U_2$ (biholomorphic) $\iff \exists$CR diffeomorphism $\partial U_1 \cong \partial U_2$.

Proof.

Uses Bergman kernel expansion. Given $U = \{\rho < 0\}$, $d\rho|_{\partial U} \neq 0$ (boundary defining function). Strongly pseudoconvex: ($\partial\bar{\partial}\rho$ positive def.). Then

$$\begin{aligned}\Pi_U(z,z) &= a(z)\,\rho^{-n-1} + b(z)\ln(-\rho)\\ &\sim \sum_{j=0}^{\infty} a_j(x)\,\rho^{-n-1+j} + \sum_{j=0}^{\infty} b_j(x)\,\rho^j \ln(-\rho), \quad \text{as } \rho \to 0,\end{aligned}$$

for $a(z), b(z) \in C^\infty(\overline{U})$, $z = (x, \rho)$ local coord near boundary. Study geodesic flow for Bergman metric near boundary to obtain boundary extension. □

Szegő parametrix

Interesting to describe the singularities of its Schwartz kernel.

Theorem (Boutet de Monvel-Sjöstrand '75)

Let $U \subset \mathbb{C}^n$ be strongly pseudoconvex domain. Then its Bergman kernel is an FIO

$$\Pi_U(x,y) = \int_0^\infty dt e^{it\Psi(x,y)} a(t;x,y)$$

Phase: $\Psi(x,x) = \frac{1}{i}\rho(x)$, $\overline{\Psi(x,y)} = \Psi(y,x)$, *and* $d_x\Psi, d_y\Psi = O(|x-y|^\infty)$
Amplitude: $a(t;x,y) \in S^n_{t,cl}$, $a \sim \sum_{j=0}^\infty t^{n-j} a_j(x,y)$

Bergman asymptotics

Their proof goes by passage to boundary $X^{2n-1} = \partial U$.

Has *CR structure*:

$T^{1,0}X = T^{1,0}\mathbb{C}^n \cap T_{\mathbb{C}}X$ ($n-1$ dimnl. integrable distribution)
$\partial_b : C^\infty(X) \to C^\infty\left(T^{1,0}X\right)$, $\partial_b f = d_X f|_{T^{1,0}X}$

Boutet de Monvel-Sjöstrand construct a parametrix for Szegő kernel Π_b (projector onto $\ker \partial_b$) and use the relation

$$\Pi_U \text{"} = \text{"} P\left(P^* P\right)^{-1} \Pi_b^0 P$$

where $P =$ Poisson operator solving Dirichlet problem.

Greens function, pointwise bounds

Consider $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type
$(HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]$ bracket generating).
Let maximum number of brackets be $r := \max_{x \in X} r(x)$.

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Theorem (Christ '89)

Let $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type. Assume $\bar{\partial}_b$ has closed range. Near any point $x' \in X$ of type $r\left(x'\right)$ there exists coordinates $\left(x_1, x_2, x_3\right)$ centered at x' such that

$$\left|\partial_x^\alpha \Pi\left(x,0\right)\right| \leq C_\alpha \left[d^H\left(x,0\right)\right]^{-2-r-\alpha_1-\alpha_2-r\alpha_3}$$

$$d^H\left(x,0\right) = |x_1| + |x_2| + |x_3|^{1/r(x')}.$$

($d^H\left(x,0\right)$ is equivalent to the sub-Riemannian CC distance between $x, 0$).

Similar bounds for boundaries of weakly pseudoconvex finite type domains in $\mathbb{C}^2$: McNeal '89, Nagel-Rosay-Stein-Wainger '89

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Theorem (Hsiao-S. '20)

Let $\left(X^3, T^{1,0}X\right)$ *weakly pseudoconvex, finite type. Assume* $\bar{\partial}_b$ *has closed range. Near any point* $x' \in X$ *of type* $r\left(x'\right)$ *there exists coordinates* $\left(x_1, x_2, x_3\right)$ *centered at* x' *such that*

$$\Pi\left(x, 0\right) = \int_0^\infty dt\, e^{itx_3} b\left(x; t\right) + C^\infty\left(X\right)$$

where $b \sim t^{\frac{2}{r}} \left[\sum_{j=0}^{\infty} t^{-\frac{j}{r}} b_j\left(t^{\frac{1}{r}} x_1, t^{\frac{1}{r}} x_2\right)\right] \in S^{\frac{2}{r}}_{\frac{1}{r}, cl}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$, $b_j \in \mathcal{S}\left(\mathbb{R}^2\right)$.

Christ estimates are equivalent to $b \in S^{\frac{2}{r}}_{\frac{1}{r}}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$
(i.e. $\partial_t^k \partial_x^\alpha b = O\left(t^{m-k+\frac{|\alpha|}{r}}\right)$ without classical expansion).

Bergman asymptotics

Specializations:

Theorem (Hsiao-S. '20)

Let $X^3 = \partial U$ *be boundary of weakly pseudoconvex, finite type domain* $U = \{\rho < 0\} \subset \mathbb{C}^2$. *For any point* $x' \in X = \partial U$ *on the boundary, of type* $r = r(x')$, *the Bergman kernel satisfies the asymptotics*

$$\Pi_U(z,z) \sim \sum_{j=0}^{\infty} \frac{1}{(-\rho)^{2+\frac{2}{r}-\frac{1}{r}j}} a_j + \sum_{j=0}^{\infty} b_j (-\rho)^j \log(-\rho),$$

as $z \to x'$ *for some set of reals* a_j, b_j *with* $a_0 > 0$.

Fefferman '74 (strongly pseudoconvex case), D'Angelo '78 (ellipsoids), Boas-Straube-Yu '95 (h-extendible/semiregular domains), Kamimoto '98, '04 (tube domains, toric domains)...

Canonical metrics

Mok-Yau '83: $U \subset \mathbb{C}^n$pseudoconvex admits a Kahler Einstein metric

$$g^{U,\mathrm{KE}} = \partial\bar{\partial} \ln u$$

u solves Monge Ampere (MA) type equation

Lee Melrose '82: $U \subset \mathbb{C}^n$ strongly pseudoconvex. MA solution u has full boundary asymptotic expansion (with leading term $u \sim \rho$).

Question (Yau '82): For which $U \subset \mathbb{C}^n$ is $g^{U,\mathrm{Bergman}} = g^{U,\mathrm{KE}}$

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Theorem (Fu-Wong '97, Huang-Xiao '2018)

$U \subset \mathbb{C}^n$ strongly pseudoconvex. If $g^{U,Bergman} = g^{U,KE}$ then $U = \mathbb{D}^n$.

Application of new asymptotics

Theorem (Xiao-S., in progress)

$U \subset \mathbb{C}^2$ weakly pseudoconvex & finite-type. If $g^{U,Bergman} = g^{U,KE}$ then $U = \mathbb{D}^2$.

Thank you.