Properties of fractional p-Laplace equations with nonlinear and sign-changing potential

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Outlines

- The fractional p-Laplace equations
- Background and Motivation
- Results and Discussion
- Brief Proof of Main results

The fractional p-Laplace equations

The equations involving the fractional p-Laplacian with nonlinear and sign changing potential,

$$\begin{cases} (-\Delta)_p^s u(x) + a(x)|u|^{p-2}u(x) = f(u(x)), & in \;\; \Omega \\ u > 0, & \text{in } \; \Omega \\ u = 0, & \text{on } \; \Omega^c \end{cases} \tag{1}$$

where Ω is either a bounded or an unbounded domain.

The fractional p-Laplace equations

The fractional p-Laplacian is a non-local and nonlinear operator defined as follows

$$(-\Delta)_p^s u(x) := C_{n,s,p} P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy$$
$$= C_{n,s,p} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy \qquad (2)$$

where $0 < s < 1$, $2 < p < \infty$, the constant $C_{n,s,p}$ depends on n, s, and p, and $P.V.$ is the Cauchy principal value.

The fractional p-Laplace equations

The function $a(x)$ in the potential may change sign here. Denote

$$\mathscr{L}_p u := (-\Delta)_p^s u + a(x)|u|^{p-2}u, \tag{3}$$

which is a positive definite operator, i.e. the first eigenvalue $\lambda_{1,sp}(a)$ of $\mathscr{L}_p$ is positive in bounded domains.

The fractional p-Laplace equations

According to the definition and existence of the first eigenvalue of $(-\Delta)^s_p$, here

$$\lambda_{1,sp}(a) := \inf\{\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dydx + \int_\Omega a(x)|u|^p dx : \|u\|_{L^p} = 1\} \tag{4}$$

The fractional p-Laplace equations

In order to make the integral in

$$(-\Delta)_p^s u(x) := C_{n,s,p} P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy$$

well defined, we require that

$$u \in C_{loc}^{1,1}(\Omega) \cap \mathcal{L}_{sp},$$

with

$$\mathcal{L}_{sp} := \{u \in L_{loc}^{p-1} | \int_{\mathbb{R}^n} \frac{|u(x)|^{p-1}}{1+|x|^{n+sp}} dx < \infty\}.$$

The fractional p-Laplace equations

Remark

- *In particular, when* $p = 2$, $(-\Delta)_p^s$ *becomes the fractional Laplacian* $(-\Delta)^s$.
- *It is showed that as* $s \to 1$, *the fractional p-Laplacian converges to the regular p-Laplacian as follows*

$$(-\Delta)_p^s u(x) \to -\Delta_p u(x) := -div(|\nabla u(x)|^{p-2} \nabla u(x)).$$

The existence of the equations involving $(-\Delta)_p^s$

- In 2016, Mosconi and his cooperators obtain nontrivial solutions to the Brezis-Nirenberg problem for the fractional p-Laplace equation.
- In 2016, Radulescu and his cooperators investigate the existence of weak solutions for a perturbed nonlinear elliptic equation driven by $(-\Delta)_p^s$, where the potential function is positive.
- In 2016, Ambrosio and his cooperators prove the existence of infinitely many weak solutions for the equation involving $\mathscr{L}_p$ with the sign changing potential $a(x)$.

The existence of the equations involving $(-\Delta)_p^s$

The Existence

Based on these work, the fractional p-Laplace equation (1)

$$\begin{cases} (-\Delta)_p^s u(x) + a(x)|u|^{p-2}u(x) = f(u(x)), & in\ \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \Omega^c \end{cases}$$

holds the existence of positive solutions.

The regularity for fully nonlinear integro-differential operators

- In 2014 Iannizzotto and his cooperators develop the global Hölder regularity (up to the boundary) for fractional p-Laplace equations.
- In 2018, Brascoand and his cooperators give the Hölder estimates for fractional p-Laplace equations in bounded domains.
- In 2019, Caffarelli and Silvestre obtain the Hölder regularity results for nonlinear integro-differential equations.

The regularity for fully nonlinear integro-differential operators

The regularity

These results ensure the regularity of our problem.

The moving plane method

- The method of moving plane is initially invented by Alexanderoff in the early 1950s.
- It is developed by Serrin in 1971, Gidas, Ni and Nirenberg in 1979 and 1981 , Caffarelli, Gidas and Spruck in 1991.
- In 1991, Chen and his cooperators apply this method for various semi-linear equations involving the fractional Laplacian to obtain symmetry, monotonicity and non-existence of solutions.
- In 2018, Chen and Li introduce maximum principles and use moving plane method to conclude the symmetry of the fractional p-Laplace equation both in bounded domains and the whole space.

The moving plane method

The moving plane method

Here we apply the moving plane method to explore that the solution of the equation (1)

$$\begin{cases} (-\Delta)_p^s u(x) + a(x)|u|^{p-2}u(x) = f(u(x)), & in \ \Omega \\ u > 0, & \text{in } \ \Omega \\ u = 0, & \text{on } \ \Omega^c \end{cases}$$

is radially symmetric in bounded domains.

The sliding method

- The sliding method is similar to the moving plane method, which can be applied for the equations without decaying conditions at infinity.
- The classical sliding method is invented by Berestycki and Nirenberg in 1990 and 1991, which is for establishing qualitative properties of solutions for partial differential equations involving Laplacian, such as symmetry, monotonicity, and uniqueness etc.
- In 2020, Chen and Wu develop the sliding method for the fractional equations.

The sliding method

- For the equation (1)

$$\begin{cases} (-\Delta)_p^s u(x) + a(x)|u|^{p-2}u(x) = f(u(x)), & in \;\; \Omega \\ u > 0, \quad \text{in} \;\; \Omega \\ u = 0, \quad \text{on} \;\; \Omega^c \end{cases}$$

  in unbounded domains, we discover that the solution is monotone increasing in some direction by using the idea of the sliding method.
- Here the classical sliding method has been modified by a sequence of proper auxiliary functions.

Main Results

Recall

$$\mathscr{L}_p u := (-\Delta)_p^s u + a(x)|u|^{p-2}u,$$

Theorem (1)

Let Ω be a bounded domain, $u \in W^{s,p}(\Omega)$ be a weak solution of the following equation

$$\begin{cases} \mathscr{L}_p u = f(u) \geq 0, & in\ \Omega \\ u \geq 0, & on\ \Omega^c \end{cases} \tag{5}$$

then, the following assertions are equivalent.

Main Results

(i) The operator $\mathscr{L}_p$ satisfies the maximum principle (MP in short)

$$u \geq 0 \quad \text{in } \Omega.$$

(ii) The operator $\mathscr{L}_p$ satisfies the strong maximum principle (SMP in short)

$$u > 0, \;\text{ or } u \equiv 0 \quad \text{in } \Omega.$$

(iii) The first eigenvalue of $\mathscr{L}_p$ is positive, i.e $\lambda_{1,sp}(a) > 0$.

Main Results

Theorem (2)

Let $\Omega = B_1(0)$, $p > 2$ and $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{sp}$ be a positive bounded solution of (1)

$$\begin{cases} (-\Delta)^s_p u(x) + a(x)|u|^{p-2}u(x) = f(u(x)), & in \ \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \Omega^c \end{cases}$$

Let the potential $a(x) \in L^\infty(\Omega)$, $\frac{df}{du}$ exist and be non-positive. Then the solution $u(x)$ is radially symmetric about the origin in Ω.

Main Results

Corollary

Let $\Omega = B_1(0)$, $p > 2$ *and* $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{sp}$ *be a bounded solution of* (1). *Let the potential* $a(x) \in L^{\infty}(B_1(0))$, *and* $a(x) > 0$, f *satisfies the condition as in above Theorem. Then the solution* $u(x)$ *is radially symmetric about the origin in* $B_1(0)$.

Main Results

Theorem (3)

Let Ω be an unbounded domain in x_n-direction, $p > 2$, $0 < s < \frac{1}{p}$. Let the solution $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{sp}$ be a positive bounded solution of (1) with the upper bound $\mu > 0$. Let the potential $a(x)$ satisfy the following condition.

(A) For any $x \in \Omega$, $|a(x)| \le c_0 f(u(x))$, where $0 < c_0 \le \frac{1}{4\mu^{p-1}}$.

Here $f(u)$ is continuous w.r.t. u and satisfies the following conditions when $0 < t_0 < t_1 < \mu$.

Main Results

(F_1) The $f(u) > 0$ if $u \in (0, \mu)$, and the $f(u) \leq 0$ if $u(x) \geq \mu$.

(F_2) The $f(u) \geq \delta_0 u$ on $[0, t_0]$ for some $\delta_0 > 0$.

(F_3) For $u \in (t_1, \mu)$, $f(u)$ is non-increasing and its decay rate is as follows

$$|\frac{df}{du}| \geq c_0[\max_{u \in (t_1,\mu)} f(u)] \cdot \frac{dG}{du}$$

where $G(u) := |u|^{p-2}u$, and the constant c_0 is the same as in (A).

Then the solution u is strictly monotone increasing in x_n-direction.

Main results

If the potential function is positive, the strictly monotone properties of the solution is also valid under more general conditions as in the following Corollary.

Corollary

Let the domain Ω be unbounded, $p > 2$, $0 < s < \frac{1}{p}$. Let $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{sp}$ be a positive bounded solution of (1), with the upper bound $\mu > 0$. Let the potential function $a(x)$ be positive, $f(u(x))$ be continuous w.r.t. u and satisfy the conditions (F_1)-(F_2) as in Theorem(3), with the following condition.

(F_3') *For $u \in (t_1, \mu)$, $f(u)$ is non-increasing.*

Then the solution u is strictly monotone increasing in x_n-direction.

Discussion

Recall

(F_1) The $f(u) > 0$ if $u \in (0, \mu)$, and the $f(u) \leq 0$ if $u(x) \geq \mu$.

(F_2) The $f(u) \geq \delta_0 u$ on $[0, t_0]$ for some $\delta_0 > 0$.

Then

- The conditions (F_1) and (F_2) of the inhomogeneous term $f(u)$ in (1) mainly follows the references by Liu in 2019 and by Chen in 2020.
- The intuitive idea of the conditions (F_1) and (F_2) comes from the following the De Giorgi Conjecture.

Discussion

Conjecture

If u is a solution of

$$-\Delta u = u - u^3,$$

such that

$$|u| \leq 1 \text{ in } \mathbb{R}^n, \quad \lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1 \ \text{ for all } \ x' \in \mathbb{R}^{n-1},$$

and $\frac{\partial u}{\partial x_n} > 0$, then there exists a vector $\mathbf{a} \in \mathbb{R}^{n-1}$ and a function $u_1 : \mathbb{R} \to \mathbb{R}$ such that

$$u(x', x_n) = u_1(\mathbf{a} \cdot x' + x_n).$$

Discussion

(F_1) The $f(u) > 0$ if $u \in (0, \mu)$, and the $f(u) \leq 0$ if $u(x) \geq \mu$.

(F_2) The $f(u) \geq \delta_0 u$ on $[0, t_0]$ for some $\delta_0 > 0$.

(F_3) For $u \in (t_1, \mu)$, $f(u)$ is non-increasing and its decay rate is as follows

$$|\frac{df}{du}| \geq c_0[\max_{u\in(t_1,\mu)} f(u)] \cdot \frac{dG}{du}$$

where $G(u) := |u|^{p-2}u$, and the constant c_0 is the same as in (A).

The condition (F_3) is consistent with (F_1) and (F_2).

Example

Let $f(u) = u^{p-1} - u^{p+1}$ for $p > 2$, we can easily verify that the condition (F_3) is satisfied when $c_0 < 1$, furthermore the condition (A) is also natural.

Discussion

Recall

(A) For any $x \in \Omega$, $|a(x)| \leq c_0 f(u(x))$, where $0 < c_0 \leq \frac{1}{4\mu^{p-1}}$.

- Comparing the equations involving fractional p-Laplacian in the references studied before with the equation (1), we can discover that the sign-changing potential function take little influence of the solution in bounded domains.
- However, the sign-changing potential in unbounded domains breaks down the analysis process in the classical sliding method. Here we build up the relationship between the inhomogeneous term and the sign-changing potential, as the condition (A). This relationship could overcome the difficulty caused by the sign-changing potential.

Discussion

Recall

(A) For any $x \in \Omega$, $|a(x)| \leq c_0 f(u(x))$, where $0 < c_0 \leq \frac{1}{4\mu^{p-1}}$.

- The condition (A) in Theorem 3 gives that the problem (1) coincides with the De Giorgi Conjecture.
- The equation (1) shares the analogous properties with the corresponding asymptotic equations in the work by Liu in 2019.

Discussion

- According to Theorem 2 and Theorem 3, we may discover that the properties of the solutions of the same equation (1) in bounded and unbounded domains are different.
- The fact that the geometry of domains determines the property of the equation (1) is mainly caused by the non-locality of the fractional p-Laplacian.

Proof of Theorem 2

General Idea

For bounded domain case, we will apply the idea of the moving plane method to give the symmetry of the problem.

Proof of Theorem 2

First, we recall an inequality for the preparation of the proof.

Lemma

For $G(t) = |t|^{p-2}t$, it is well known that by the mean value theorem, we have

$$G(t_2) - G(t_1) = G'(\xi)(t_2 - t_1).$$

Then there exists a constant $c_1 > 0$ such that

$$|\xi| \geq c_1 \max\{|t_1|, |t_2|\}.$$

Proof of Theorem 2

Let

$$\begin{aligned}
T_\lambda &:= \{x \in \mathbb{R}^n | x_1 = \lambda, \textit{ for some } \lambda \in \mathbb{R}\}, \\
\Sigma_\lambda &:= \{x \in \mathbb{R}^n | x_1 < \lambda\}, \\
x^\lambda &:= (2\lambda - x_1, x_2, \ldots, x_n), \\
\omega_\lambda(x) &:= u(x^\lambda) - u(x), \\
\Omega_\lambda &:= \Sigma_\lambda \cap B_1(0).
\end{aligned} \tag{6}$$

where x^λ is the reflection of x about the T_λ.

Proof of Theorem 2

We will carry out the proof in two steps.

- Step 1. We will show that for $\lambda \to -1$, we have

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda, \tag{7}$$

  which is the starting point for moving plane method.

- Step 2. The move the plane T_λ toward right to its limiting position as long as (7) holds. We will show that the limiting position is $\lambda = 0$.

Radially symmetry

Since the arbitrariness of direction, we obtain that the solution u is radially symmetric about the origin.

Proof of Theorem 3

General Idea

For unbounded domain case, we will apply the idea of the sliding method and construct the auxiliary functions, the domain of which changes along with the maximizing sequence.

Proof of Theorem 3

First, we prove that there exists a positive lower bound of the solution u.

Lemma

Let u satisfy conditions in the following Proposition, then there exists $\widehat{c}$, $R_0 > 0$, where R_0 depends only on n and δ_0 in the condition (F_2) such that

$$u(x) > \widehat{c} \quad as \quad dist(x, \partial\Omega) > R_0.$$

Proof of Theorem 3

By the above Lemma, we have

Proposition

Let $u \in C^{1,1}_{loc}(\Omega) \cap \mathcal{L}_{sp}$ solve the equation (1), the functions $a(x)$ and $f(u)$ satisfy the conditions in Theorem 3. Then $u(x) \to \mu$ as $x_n \to \infty$.

Proof of Theorem 3

Let's choose an auxiliary function $\phi \in C_0^\infty(\mathbb{R}^n)$ as follows,

$$\phi(x) = \begin{cases} ce^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{8}$$

where $c > 0$ with $\phi(0) = 1$. For $z \in \Omega$, take $R > 0$ such that $B_R(z) \subset \Omega$.

Proof of Theorem 3

Define

$$\phi_R(x) := \phi(\frac{x-z}{R}), \;\; \phi_{\varepsilon,R}(x) := \varepsilon\phi(\frac{x-z}{R}).$$

It's obvious that $\phi_{\varepsilon,R}(z) = \max_{x\in B_R(z)} \phi_{\varepsilon,R}(x) = \varepsilon$. Then we have that

$$\begin{cases} (-\Delta)_p^s \phi_{\varepsilon,R}(x) + a(x)|\phi_{\varepsilon,R}|^{p-2}\phi_{\varepsilon,R}(x) \leq [\frac{C}{R^{sp}} + |a(x)|]\varepsilon^{p-1}, \text{ in } B_R(z) \\ \phi_{\varepsilon,R}(x) = 0, \text{ on } B_R^c(z) \end{cases} \tag{9}$$

where C is a positive constant.

Proof of Theorem 3

Remark

In the proof of the above Proposition, the condition of the decay rate of the inhomogeneous term f has not been used. Hence in the Proposition, we can relax the condition (F_3) in Theorem (3)to the condition (F_3') as follows.

(F_3') *For $u \in (t_1, \mu)$, $f(u)$ is non-increasing.*

Proof of Theorem 3

Remark

The Proposition indicates that there exists a maximizing sequence of the solution of the equation (1) *near infinity, which is the most important property of the sliding process.*

Proof of Theorem 3

For $\tau \geq 0$, denote

$$u_\tau(x) := u(x + \tau e_n), \;\; \omega_\tau(x) := u(x) - u_\tau(x),$$

where $e_n = (0, 0, \cdots, 1)$.

Proof of Theorem 3

We will carry out the proof in two steps.

- Step 1. We will show that for $\tau > 0$ sufficiently large, we have

$$\omega_\tau(x) \leq 0, \quad \text{for } x \in \mathbb{R}^n. \tag{10}$$

  This provides the starting point for the sliding method.

- Step 2. Decrease τ continuously as long as (10) is valid, we will prove that for any $\tau > 0$,

$$\omega_\tau(x) \leq 0, \quad \text{for } x \in \mathbb{R}^n. \tag{11}$$

Proof of Theorem 3

Define

$$\tau_0 := \inf\{\tau > 0 | \omega_\tau(x) \le 0, \forall x \in \mathbb{R}^n\},$$

we claim that

$$\tau_0 = 0. \tag{12}$$

Idea of the proof for Step 2

The claim (12) is equivalent to (11), which implies that the solution u is monotone increasing in x_n- direction.

Thank you for your attention!